Stress-energy tensor—25 Mar 2010

- Definition of the stress energy tensor $T^{\mu\nu}$
- Stress energy tensor of a particles
- Stress energy tensor of a perfect gas
- Energy and momentum conservation $\nabla_\nu T^{\mu\nu} = 0$
- "Derivation" of Einstein's field equation

◄ | ►

Example: Curvature tensors for the surface of a 2-d sphere

The metric is

$$ds^2 = a^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

The nonzero parts of the Christoffel symbol are

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta$$

The Riemann-Christoffel tensor is in general

$$R^{\sigma}{}_{\gamma\alpha\beta} = \frac{\partial}{\partial x^{\alpha}}\Gamma^{\sigma}{}_{\gamma\beta} - \frac{\partial}{\partial x^{\beta}}\Gamma^{\sigma}{}_{\gamma\alpha} + \Gamma^{\sigma}{}_{\alpha\epsilon}\Gamma^{\epsilon}{}_{\gamma\beta} - \Gamma^{\sigma}{}_{\beta\epsilon}\Gamma^{\epsilon}{}_{\gamma\alpha}$$

Q: Compute the component $R^{\theta}{}_{\phi\theta\phi}$

$$R^{\theta}{}_{\phi\theta\phi} = \ldots = \sin^2\theta$$

Q: Compute the component

$$R^{\theta\phi}{}_{\theta\phi}$$

Q: Compute the Ricci tensor. Answer:

$$R^{\theta}{}_{\theta} = R^{\phi}{}_{\phi} = a^{-2}$$

$$R^{\theta}{}_{\phi} = R^{\phi}{}_{\theta} = 0$$

Q: Compute the curvature scalar R

The metric is

$$ds^2 = a^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

The nonzero parts of the Christoffel symbol are

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta$$

The Riemann-Christoffel tensor is in general

$$R^{\sigma}{}_{\gamma\alpha\beta} = \frac{\partial}{\partial x^{\alpha}}\Gamma^{\sigma}{}_{\gamma\beta} - \frac{\partial}{\partial x^{\beta}}\Gamma^{\sigma}{}_{\gamma\alpha} + \Gamma^{\sigma}{}_{\alpha\epsilon}\Gamma^{\epsilon}{}_{\gamma\beta} - \Gamma^{\sigma}{}_{\beta\epsilon}\Gamma^{\epsilon}{}_{\gamma\alpha}$$

Q: Compute the component $R^{\theta}{}_{\phi\theta\phi}$.

$$R^{\theta}{}_{\phi\theta\phi} = \frac{\partial}{\partial x^{\theta}}\Gamma^{\theta}{}_{\phi\phi} - \frac{\partial}{\partial x^{\phi}}\Gamma^{\theta}{}_{\phi\theta} + \Gamma^{\theta}{}_{\theta\epsilon}\Gamma^{\epsilon}{}_{\phi\phi} - \Gamma^{\theta}{}_{\phi\epsilon}\Gamma^{\epsilon}{}_{\phi\theta}$$

$$= \frac{\partial}{\partial x^{\theta}} -\sin\theta\cos\theta - \frac{\partial}{\partial x^{\phi}}\Gamma^{\theta}{}_{\phi\theta}(\text{zero}) + \Gamma^{\theta}{}_{\theta\phi}(\text{zero})\,\Gamma^{\phi}{}_{\phi\phi} - \Gamma^{\theta}{}_{\phi\theta}(\text{zero})\,\Gamma^{\theta}{}_{\phi\theta} - \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\phi\theta}$$

$$= \frac{\partial}{\partial x^{\theta}} -\sin\theta\cos\theta - (-\sin\theta\cos\theta)\cot\theta$$

$$= \sin^2\theta + \cos^2\theta - \cos^2\theta$$

$$= \sin^2\theta$$

Q: Compute the component

$$R^{\theta\phi}{}_{\theta\phi}$$

Q: Compute the Ricci tensor. Answer:

$$R^{\theta}{}_{\theta} = R^{\phi}{}_{\phi} = a^{-2}$$

$$R^{\theta}{}_{\phi} = R^{\phi}{}_{\theta} = 0$$

Q: Compute the curvature scalar R

Stress-energy tensor without gravity

Definition of the stress-energy tensor $T^{\alpha\beta}$.
1) Let u^β be the 4-velocity of the observer. Then
$$T^\alpha{}_\beta\, u^\beta = T_\beta{}^\alpha\, u^\beta = -d\; p^\alpha / d \text{ volume}$$
is the density of 4-momentum. $-T^\alpha{}_\beta\, u^\beta$ dx dy dz is the 4-momentum in a box.
Q: Let n^α be a unit vector. What is $T_{\alpha\beta}\, u^\beta\, n^\alpha$?
2) Let i and j be indices in space. $T_{ij} = T_{ji}$ is the force in the i direction on a unit surface perpendicular to the j direction. It is also the force in the j direction on a unit surface perpendicular to the i direction.
Q: What is T_{xx}?

Stress energy tensor for a swarm of particles.
The particles have mass m and 4-velocity u. Their momentum is $p = m\, u$. There are n particles per unit volume in the frame of the particles.

In a frame in which the particles are moving, the flux of particles is
$$s = n\, u.$$
The x component of s is the number of particles per second crossing a unit area perpendicular to the x-direction.
Q: What is s^0?
Q: $s^0 = n\left(1 - v^2\right)^{-1/2}$. What is the reason for the factor $\left(1 - v^2\right)^{-1/2}$?

Since each particle carries momentum p, the density of 4-momentum is
$$T^{\alpha 0} = p^\alpha\, s^0$$
$$= \; m\, u^\alpha\, n\, u^0$$
and the flux of 4-momentum is
$$T^{\alpha i} = p^\alpha\, s^\iota$$
$$= \; m\, u^\alpha\, n\, u^i$$
All together,
$$T^{\alpha\beta} = m\, n\, u^\alpha\, u^\beta.$$

Stress energy tensor for a perfect gas. Consider the frame in which the gas is at rest.

The T^{00} term is the sum of $m\, n\, u^0\, u^0$. $m\, u^0$ is the mass-energy of the particle. $n\, u^0$is the number density. The product is the mass-energy density ρ.

The T^{xx} term is the sum of $m\, n\, u^x\, u^x$. What is this?

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

where ρ is the mass-energy density and P is the pressure.
Q: There are gas particles moving in the x and y directions. Why don't they transfer momentum across the y-z plane in the y direction?

In some other frame, let u be the 4-velocity of the gas. In this frame,

$$T^{\alpha\beta} = (\rho + P)\, u^\alpha\, u^\beta + P\, \eta^{\alpha\beta}.$$

ρ and P are the mass-energy density and pressure in the rest frame; they are scalars. This is clearly a tensor.
Check that it is correct in the frame in which the fluid is at rest: $u^\alpha = (1, 0, 0, 0)$.

$$T^{00} = (\rho + P)\,(1)\,(1) + P(-1) = \rho.$$

$$T^{11} = (\rho + P)\,(0)\,(0) + P(1) = P.$$

Conservation of energy and momentum

The divergence of the stress-energy tensor is 0

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta} = \frac{\partial}{\partial t} T^{\alpha 0} + \frac{\partial}{\partial x} T^{\alpha \mathrm{x}} + \frac{\partial}{\partial y} T^{\alpha \mathrm{y}} + \frac{\partial}{\partial z} T^{\alpha \mathrm{z}} = 0$$

Interpretation: Integrate this inside a fixed 3-d box

$$\int\left(\frac{\partial}{\partial t} T^{\alpha 0} + \frac{\partial}{\partial x} T^{\alpha \mathrm{x}} + \frac{\partial}{\partial y} T^{\alpha \mathrm{y}} + \frac{\partial}{\partial z} T^{\alpha \mathrm{z}}\right) \mathrm{dx\,dy\,dz}$$

$$= \frac{\partial}{\partial t} \int T^{\alpha 0}\, \mathrm{dx\,dy\,dz} + \int T^{\alpha \mathrm{x}}(x + \mathrm{dx})\, \mathrm{dy\,dz} - \int T^{\alpha \mathrm{x}}(x)\, \mathrm{dy\,dz} + \ldots = 0$$

We said that $T^{\alpha 0}$is the density of the α component of the 4 momentum, and $T^{\alpha \mathrm{x}}$ is the flux of the the α component of the 4 momentum in the x direction $\left(d\ p^\alpha / s \middle/ m^2\right)$

The change in the amount of p^α inside the box + the amount going out through the surfaces is zero.

Stress-energy tensor with gravity

Q: What is the rule for including gravity?

Energy-momentum conservation is

$$\nabla_\alpha T^{\alpha\beta} = 0.$$

For a perfect gas,

$$T^{\alpha\beta} = (P + \rho)\, u^\alpha\, u^\beta + P\, g^{\alpha\beta}.$$

◄ | ►

The cosmological constant

Einstein's argument:
The equivalence principle says there exists a frame n which the effects of gravity vanish. In that frame the metric is $\eta^{\alpha\beta}$ and first derivatives of $\eta^{\alpha\beta}$vanish. (The first derivatives give nonzero Christoffel symbols.) $\frac{\partial}{\partial x^{\alpha}} \eta^{\alpha\beta} = 0$. The transformation of the zero tensor is zero. Therefore

$$\nabla_\alpha g^{\alpha\beta} = 0.$$

A stress-energy tensor having the form

$$T^{\alpha\beta} = -\Lambda\, g^{\alpha\beta},$$

where Λ is a constant, is conserved. I can think of no reason why such a such a stress-energy tensor cannot exist.

What pressure and mass-energy density does this imply? In the frame in which the gas is at rest,

$$T^{\alpha\beta} = -\Lambda\, g^{\alpha\beta} = (P + \rho)\, u^\alpha\, u^\beta + P\, g^{\alpha\beta}$$

becomes

$$\begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & -\Lambda & 0 & 0 \\ 0 & 0 & -\Lambda & 0 \\ 0 & 0 & 0 & -\Lambda \end{pmatrix} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

The pressure $P = -\rho$.

◄ | ►

"Derivation" of Einstein's field equation

The plan is to write

measure of curvature = source of gravity.

0. Einstein & Grossman, Z. Math. Physik, 62, 225, (1913) the mathematics of curvature.

1. A guess for the source of gravity is the stress-energy tensor. In the limit of slow speeds, the stress-energy tensor is

$$T^{\alpha\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \longrightarrow \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Q: Why does $P \to 0$ for slow speeds?
The mass density is the source of gravity for Newton's gravity.

2. We have many choices for a measure of curvature.
Q: What are the choices?
Riemann-Christoffel curvature tensor (rank 4), Ricci tensor (rank 2), metric tensor (rank 2), and curvatrue scalar.
For the LHS to equal the RHS, we have to use the same rank for both.
Q: What rank 2 tensor is a measure of curvature?

3. For slowly moving particles, we reasoned using the equivalence prnciple.
The equation of motion is

$$\frac{d u^i}{d\tau} + \Gamma^i_{\nu\alpha} u^\nu u^\alpha = 0$$

The biggest term is u^0. The equation of motion is

$$\frac{d u^i}{d\tau} + \Gamma^i_{00} = 0.$$

We computed Γ to find

$$\frac{du^i}{dt} + \frac{1}{2} \frac{\partial g_{00}}{\partial x^i} = 0.$$

Newtons' equation is

$$\frac{du^i}{dt} + \frac{\partial \phi}{\partial x^i} = 0$$

Therefore

$$g_{00} = -(1 + 2\,\phi).$$

We know $\nabla^2 \phi = 4\,\pi\,\mathrm{G}\rho$.
Therefore the constant is $-8\,\pi\,G$.

4. Here was Einstein's guess:

$$R^{\mu\nu} = -8\pi G T^{\mu\nu}.$$

5. There was a big problem. Energy and momentum conservation means $\nabla_\nu T^{\mu\nu} = 0$. However $\nabla_\nu R^{\mu\nu} \neq 0$. Einstein tried several patches.

6. We know about Bianchi's identity (that carrying a vector around the 6 faces of a cube yields 0), but Einstein didn't.
Q: What is Bianchi's identity? (class of 23 Mar) What should the LHS be?

7. Success. 25 Nov 1915, Prussian Acad. Wissen, p8

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -8\pi G T^{\mu\nu}$$

◄ | ►

Newtonian limit of the equation of motion (From 4 March with error fixed)

Consider the case of a ball and the stationary earth. Nonrelativistic limit $v \ll 1$: The 4-velocity

$$\frac{dx}{d\tau} = \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau}\right) \approx (1, 0, 0, 0).$$

The equation of motion

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\nu\alpha} \frac{dx^\nu}{d\tau} \frac{dx^\alpha}{d\tau} = 0$$

is then

$$\frac{d^2 x^1}{d\tau^2} + \Gamma^1_{\nu\alpha} \frac{dx^\nu}{d\tau} \frac{dx^\alpha}{d\tau} = 0.$$

Since $\frac{dx^1}{d\tau}$ is of order v, whereas $\frac{dx^0}{d\tau}$ is of order 1, the biggest term involves Γ^λ_{00}

$$\Gamma^\sigma_{00} = \frac{1}{2} g^{\nu\sigma}\left(g_{0\,\nu,0} + g_{0\,\nu,0} - g_{00,\nu}\right)$$

Since earth is stationary, time derivatives of the metric are zero.

$$\Gamma^\sigma_{00} = -\frac{1}{2} g^{\nu\sigma} g_{00,\nu}$$

Assume the metric is

$$\begin{pmatrix} -1 - h_{00} & 0 & 0 & 0 \\ 0 & 1 - h_{11} & 0 & 0 \\ 0 & 0 & 1 - h_{22} & 0 \\ 0 & 0 & 0 & 1 - h_{33} \end{pmatrix}$$

$$\Gamma^0_{00} = -\frac{1}{2} g^{\nu 0} g_{00,\nu} = 0$$

$$\Gamma^1_{00} = -\frac{1}{2} g^{11} g_{00,1} = \frac{1}{2} \frac{\partial h_{00}}{\partial x^1}$$

The equation of motion is

$$\frac{d^2 x^1}{dt^2} + \frac{1}{2} \frac{\partial h_{00}}{\partial x^1} = 0.$$

I know that

$$\frac{d^2 \overline{x}}{dt^2} = -\text{grad}\,\phi$$

Therefore $h_{00} = 2\,\phi$. In the nonrelativistic limit, the tt term in the metric is

$$g_{00} = -(1 + 2\,\phi).$$

The 1 comes from the Minkowski metric. Gravity enters as twice the potential energy.