Black holes
—20 Apr 2010

- Outline
    - Hints of strangeness from our study of the orbits in the Schwarzscild metric
    - Eddington-Finkelstein coordinates for the Schwarzschild metric (§12). (today)
    - Stellar collapse (§24)
    - Observations of black holes
    - Radiation from black holes

Hints of strangeness

The Schwarzschild metric with the usual coordinate system is

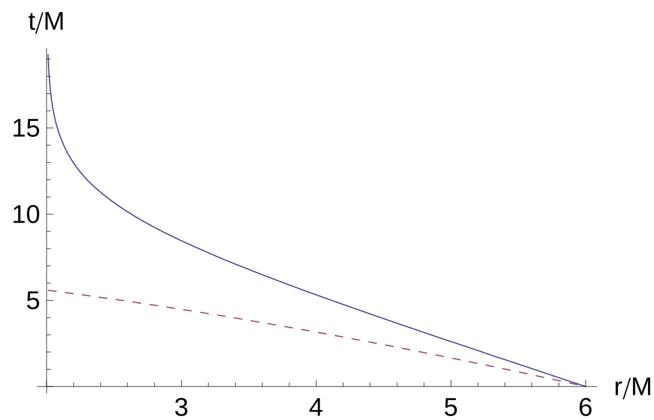
$$d s^2 = -\left(1 - \frac{2M}{r}\right) d t^2 + \left(1 - \frac{2M}{r}\right)^{-1} d r^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

- **Strangeness #1**

Q: The time coordinate is t, and the coordinates r, θ, and ϕ are spatial coordinates for all $r > 0$. True or false?

- **Strangeness #2**

In Homework 3, Problem 3, we found that for a radial orbit the proper time and coordinate time to fall radially from $r = 6\,M$.



Caption: The proper time (purple, dashed) and coordinate time (blue) to fall from $6\,M$ to r.

An observer gets reports from a guy falling radially into a black hole. One report sent at $r = 6\,M$ says, "I started by clock." The next report said, "I am passing $r = 3\,M$. The time on my watch is [garbled]."
Q: What was the report sent at $r = 3\,M$?

Q: Sagredo: "The coordinate system is singular at $r = 2\,M$. I can fix that by changing my coordinate system." Simplicio: "That might make the proper time singular." Is Simplicio's concern justified?

Eddington-Finkelstein coordinates

Consider radial light rays.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 = 0.$$

I can write

$$dt = \pm dr \left(1 - \frac{2M}{r}\right)^{-1} = \pm d\left(r + 2M \log\left|\frac{r}{2M} - 1\right|\right).$$

Let $r^* = r + 2M \log\left|\frac{r}{2M} - 1\right|$.

For incoming light rays, $dr < 0$. I choose the – sign. Then $d(t + r^*) = 0$.
For outgoing light rays, I choose the + sign. Then $d(t - r^*) = 0$.

Define a new coordinate system (v, r), where

$$\boxed{v \equiv t + r^* = t + r + 2M \log\left|\frac{r}{2M} - 1\right|}$$

This coordinate system has the property that radial incoming light rays are at v = constant.

The metric is

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}$$

There is no longer a singularity at $r = 2M$.

Consider the radial light rays. $d\theta = d\phi = 0$.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr = 0$$

One solution is $dv \neq 0$. Then

$$\frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{2M}{r}\right)$$

$$v = 2r + 4M \log\left|\frac{r}{2M} - 1\right| + \text{const}$$

At $r > 2M$, $\frac{dr}{dv} > 0$. This is an outgoing light ray.

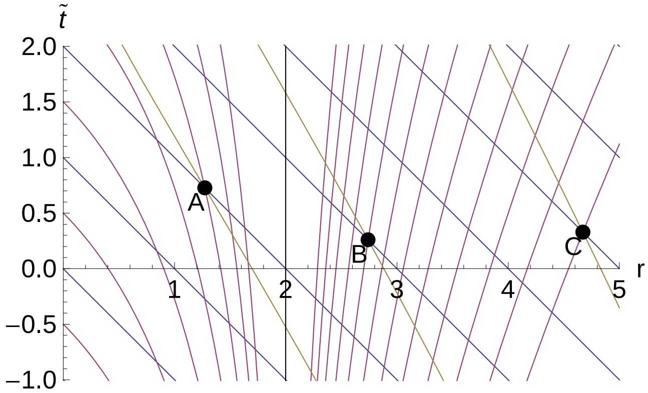
Ar $r = 2M$, $\frac{dr}{dv} = 0$. The light ray is not progressing.

At $r < 2M$, $\frac{dr}{dv} < 0$. The light ray is moving to smaller r.

The other solution is $dv = 0$. We want something to change for light rays. Define $\tilde{t} = v - r$.
For this solution, $\tilde{t} = -r + \text{const}$.These are incoming light rays.

The metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 + 4\,\frac{M}{r}\,d\tilde{t}\,dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$



Caption: Radial light rays for solution $dv = 0$ (blue) and $dv \neq 0$ (purple). Also shown are the world lines of three apples falling radially (taupe) from rest at ∞, which cross events A, B,and C. The coordinates are $(\tilde{t}, r)$

Q: At $r > 2M$, how are the light rays of the two radial solution different? Consider two radial light pulses sent out at event B.
Q: At $r < 2M$, how are the light rays of the two radial solution different? Consider two radial light pulses sent out at event A.
Q: For $r < 2M$, is dr time-like or space-like? Eg, is the separation between $(\tilde{t}, r, \theta, \phi)$ and $(\tilde{t}, r + dr, \theta, \phi)$ time-like or space-like? Same question for dt. Look at the region near event A.

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