Midterm exam
—18 Feb 2010

Problem 1

Since this metric is diagonal, there is only one term in a sum such as

$$a_\mu = g_{\mu\nu}\, a^\nu$$

$g^{\theta\theta}$ is defined as the metric for covariant vectors.

Consider a vector x that has only the θ component. Since the dot product of two vectors is the same whether one uses the covarient or contravarient forms,

$$g_{\theta\theta}\, x^\theta\, x^\theta = g^{\theta\theta}\, x_\theta\, x_\theta.$$

No sum is intended. Since the metric is diagonal, there is only a single term.
Since $x_\theta = g_{\theta\theta}\, x^\theta$,

$$g_{\theta\theta}\, x^\theta\, y^\theta = g^{\theta\theta}\, g_{\theta\theta}\, x^\theta\, g_{\theta\theta}\, y^\theta.$$

Therefore

$$g^{\theta\theta} = (g_{\theta\theta})^{-1}$$
$$= \mathrm{r}^{-2}$$

Problem 2

(a) The conserved energy is

$$p_t = E.$$

Since $p_t = g_{t\,t}\, p^t$,

$$p^t = (g_{t\,t})^{-1}\, E = -\left(1 - \frac{2M}{r}\right)^{-1} E.$$

Since the mass of a photon is zero,

$$g_{t\,t}\, p^t\, p^t + g_{r\,r}\, p^r\, p^r = 0.$$

Since the conserved energy is $p_t = g_{t\,t}\, p^t$,

$$(g_{t\,t})^{-1}\, p_t\, p_t + g_{r\,r}\, p^r\, p^r = 0.$$
$$p^r = \pm\, p_t (-g_{t\,t}\, g_{r\,r})^{-1/2}$$
$$= \pm\, p_t$$

Therefore

$$p^r = E\left(-\left(1 - \frac{2M}{r}\right)^{-1}, -1, 0, 0\right).$$

(b) The energy measured in a lab is the magnitude of the time part of the 4-momentum, which is

$$[-p^t\, p_t]^{1/2} = E\left(1 - \frac{2M}{r}\right)^{-1/2}.$$

Problem 3

(a) The gravity of earth and the sun slow clocks on earth. In addition, the earth is moving, and this slows clocks too.

(b)

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Convert[AstronomicalUnit, Kilo Meter]
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1.49598×10^8 Kilo Meter

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mEarth = 4.3*^-6; rEarth = 6.4*^3; mSun = 1.5; au = 1.5*^8; vEarth = 30. / 300*^3;
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The graviational slowing: The time elapsed for a person at r is

$$\left(1 - \frac{2M}{r}\right)^{1/2} d\,t \approx \left(1 - \frac{M}{r}\right) d\,t,$$

where $d\,t$ is the change in coordinate time, which is the same as the time elapsed for a person at ∞.
Because of the presence of earth and the sun, the observer on earth sees less time pass by

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{mEarth / rEarth, mSun / au}
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$\{6.71875 \times 10^{-10},\ 1. \times 10^{-8}\}$

Because the earth is moving, clocks on earth are slower by $\left(1 - v^2\right)^{1/2} \approx 1 - \frac{1}{2} v^2$

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1/2 vEarth^2
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$5. \times 10^{-9}$

In order of the size of the effect, the gravitational slowing of the sun is biggest, followed by the moving clock, followed by the gravitational slowing of the earth. The year for an observer at ∞ is longer than a year for an observer on earth by

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Total[%%] + %
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1.56719×10^{-8}

If you started from $u^t = \left(1 - \frac{3M}{r}\right)^{-1/2}$ on your cheat sheet:
The observer on earth measures proper time, and the observer at ∞ measures time that is the same as coordinate time.

$$u^t = \frac{d\,t}{d\,\tau}.$$

$$t_e = \tau = t / u^t$$

$$= \ t_\infty \left(1 - \frac{3M}{r}\right)^{1/2}$$

$$\approx t_\infty \left(1 + \frac{3}{2}\ \frac{M}{r}\right)$$

- **Problem 4**

(a) The effective potential is

$$V_{\text{eff}}(r) = \frac{1}{2}\ \frac{l^2}{r^2} - \frac{M}{r} - \frac{l^2 M}{r^3}$$

For an object to go in and turn around, $\frac{d\,r}{d\,\tau} = 0$. At small r, V_{eff} is negative. This can only occur if V_{eff} has a maximum and a minimum, that $\frac{d\,V_{\text{eff}}(r)}{d\,r} = 0$.

$$\frac{d\,V_{\text{eff}}(r)}{d\,r} = -\frac{l^2}{r^3} - \frac{M}{r^2} + \frac{3\,l^2 M}{r^4} = 0$$

occurs at

$$r = \frac{l^2}{2M}\left[1 \pm \left(1 - 12\,(M/l)^2\right)^{1/2}\right].$$

The larger value of r determines a circular orbit, and the smaller one determines the energy for capture.

If $(l/M)^2 < 12$, there are no real roots. There is no value for the energy for which $\mathrm{dr}/\mathrm{dt} = 0$. The object is captured.

(b) At infinity, $l = b\,v$. $\left(\frac{b}{v}\right)^2 = l_{\text{capture}}^2 = 12\,M^2$

$$A = \pi\,b^2$$
$$= \ 12\,\pi\,M^2\,v^{-2}$$

If $v = 0.001$,

$$A = 1.2\times 10^7\,\pi\,M^2.$$

Liyuan pointed out that the cross section is larger. The particle, which has $\left(e^2 - 1\right)/2 \approx 0$, falls into the black hole if the peak of $V_{\text{eff}} = 0$ at the maximum

$$r_1 = \frac{l^2}{2\,M}\left[1 - \left(1 - 12\,(M/l)^2\right)^{1/2}\right].$$

This is very messy. The answer is that this occurs at $l^2 = 16\,M$. The cross section is

$$A = 1.6\times 10^7\,\pi\,M^2$$

The algebra:

$$V_{\text{eff}}(r_1) = \frac{1}{2\,r_1^3}\left\{\frac{l^4}{4}\left[1 - \left(1 - 12\,l^{-2}\right)^{1/2}\right] - \frac{l^4}{4}\left[1 - \left(1 - 12\,l^{-2}\right)^{1/2}\right]^2 - l^2\right\}$$
$$= \ \frac{l^2}{8\,r_1^3}\left\{l^2\left[\left(1 - 12\,l^{-2}\right)^{1/2} - \left(1 - 12\,l^{-2}\right)\right] - 4\right\}$$
$$= \ \frac{l^2}{8\,r_1^3}\left\{l^2\left(1 - 12\,l^{-2}\right)^{1/2} - l^2 + 8\right\}$$

Solve: $V_{\text{eff}}(r_1) = 0$ to get

$$l^4 - 12\,l^2 = l^4 - 16\,l^2 + 64$$
$$l^2 = 64/4 = 16$$