

Physics 472 Final Exam – May 3, 2010

Total points = 30. Show all your work!

Useful relations about Angular Momentum:

$$[J_x, J_y] = i\hbar J_z \text{ and cyclic permutations.}$$

$$J^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle$$

$$J_z |j, m\rangle = \hbar m |j, m\rangle$$

$$J_+ = J_x + iJ_y$$

$$J_- = J_x - iJ_y$$

$$J_+ |j, m\rangle = \hbar \sqrt{j(j+1) - m(m+1)} |j, m+1\rangle$$

$$J_- |j, m\rangle = \hbar \sqrt{j(j+1) - m(m-1)} |j, m-1\rangle$$

Spin-1/2:

$$\vec{S} = \frac{\hbar}{2} \vec{\sigma}, \quad \text{where} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over every coefficient, e.g., for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	\dots
M	M	\dots
m_1	m_2	\dots
m_1	m_2	\dots
\vdots	\vdots	\vdots
\vdots	\vdots	\vdots
Coefficients		

$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$

$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$

$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$

$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$

$Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$

$2 \times 1/2$

5/2	5/2	3/2
+5/2	1	+3/2+3/2
+2-1/2	1/5	4/5
+1+1/2	4/5-1/5	5/2
	+1/2	3/2

$1 \times 1/2$

3/2	3/2	1/2
+3/2	1	+1/2+1/2
+1-1/2	1/3	2/3
0+1/2	2/3-1/3	-1/2-1/2
	0-1/2	2/3
	-1+1/2	1/3
	-1-1/2	-3/2
	-1	1

2×1

3	3	2
+3	+2	+2
+2	0	1/3
+1	+1	2/3
	+1	-1/3
	+1	+1
	+1	+1

$3/2 \times 1$

5/2	5/2	3/2
+5/2	1	+3/2+3/2
+3/2	0	2/5
+1/2	+1	3/5
	+1/2	-2/5
	+1/2	+1/2

1×1

2	2	1
+2	+1	+1
+1	0	1/2
0	+1	1/2
	+1	-1/2
	+1	0
	+1	0

$3/2 \times 1/2$

2	2	1
+2	+1	+1
+3/2	-1/2	1/4
+1/2	+1/2	3/4
	+1/2	-1/4
	+1/2	1/2
	+1/2	1/2
	+1/2	-1/2
	+1/2	-1/2

1×1

2	2	1
+2	+1	+1
+1	0	1/2
0	+1	1/2
	+1	-1/2
	+1	0
	+1	0

$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$

$d_{m,0}^\ell = \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m e^{-im\phi}$

$\langle j_1 j_2 m_1 m_2 | j_1 j_2 JM \rangle$

$= (-1)^{J-j_1-j_2} \langle j_2 j_1 m_2 m_1 | j_2 j_1 JM \rangle$

Non-Degenerate Perturbation Theory: $H = H^0 + \lambda H'$ $H^0 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(0)}\rangle$

$$E_n = E_n^{(0)} + \lambda \langle \psi_n^{(0)} | H' | \psi_n^{(0)} \rangle + \lambda^2 \sum_{m \neq n} \frac{|\langle \psi_m^{(0)} | H' | \psi_n^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}} + \dots$$

$$|\psi_n\rangle = |\psi_n^{(0)}\rangle + \lambda \sum_{m \neq n} |\psi_m^{(0)}\rangle \frac{\langle \psi_m^{(0)} | H' | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} + \dots$$

Degenerate Perturbation Theory: If an energy level is n -fold degenerate with respect to H^0 , then the first order energy shifts and corresponding eigenstates of H are given by the eigenvalues and eigenvectors of the $n \times n$ matrix with elements $H'_{jk} = \langle \psi_j^{(0)} | H' | \psi_k^{(0)} \rangle$, where $j, k = 1, 2, \dots, n$.

Symmetries are a powerful tool to help determine which matrix elements are zero. For example, for hydrogen eigenstates $|nlm\rangle$, $[L_z, z] = 0$ implies $(m - m') \langle n'l'm' | z | nlm \rangle = 0$, and the parity transformation $\Pi z \Pi = -z$ implies $(-1)^{l+l'} \langle n'l'm' | z | nlm \rangle = -\langle n'l'm' | z | nlm \rangle$.

Variational Method: If the ground state energy of H is E_0 , then for any normalized state $|\psi\rangle$,

$$E_0 \leq \langle \psi | H | \psi \rangle$$

Hydrogen Atom: $H = \frac{p^2}{2m} - \frac{e_2}{4\pi\epsilon_0 r}$ $E_n = \frac{-Ry}{n^2}$ where $Ry = \frac{1}{2} \frac{e^2}{4\pi\epsilon a_0} = \frac{\hbar^2}{2ma_0^2} = 13.6\text{eV}$

$$R_{10} = 2a^{-3/2} e^{-r/a} \quad R_{20} = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2} \frac{r}{a}\right) e^{-r/2a} \quad R_{21} = \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} e^{-r/2a}$$

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \quad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \cos \theta \quad Y_1^{\pm 1} = \mp \left(\frac{3}{8\pi}\right)^{1/2} \sin \theta e^{\pm i\phi}$$

Harmonic Oscillator: $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$ $E_n = \left(n + \frac{1}{2}\right) \hbar\omega$

Infinite Square Well for $0 < x < a$: $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$ $E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$

Integrals: $\int_0^\infty x^n e^{-x/a} dx = n! a^{n+1}$ $\int_0^\infty x^{2n} e^{-x^2/a^2} dx = \sqrt{\pi} \frac{(2n)!}{n!} \left(\frac{a}{2}\right)^{2n+1}$ $\int_0^\infty x^{2n+1} e^{-x^2/a^2} dx = \frac{n!}{2} a^{2n+2}$

Time-dependent perturbation theory: If a system in an eigenstate of energy E_i is subjected to a time-dependent perturbation $H'(t)$ starting at time $t = 0$, the first-order probability amplitude for a transition to an eigenstate with energy E_f is:

$$c_f(t) = \frac{-i}{\hbar} \int_0^t \langle f | H'(t') | i \rangle e^{i(E_f - E_i)t'/\hbar} dt'$$