Physics 472 – Spring 2008

Homework #12, due Friday, April 23
(Point values are in parentheses.)

Consider a spin-1/2 particle. Back in January I showed you that the spinor eigenstates of $\vec{S}\cdot\hat{n}$, where $\hat{n}$ is a unit vector lying in the x-z plane at an angle θ to the z-axis, are:
$\left|\theta_\uparrow\right\rangle = \begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix}$ and $\left|\theta_\downarrow\right\rangle = \begin{pmatrix}\sin(\theta/2)\\ -\cos(\theta/2)\end{pmatrix}$. (These are special cases of Eqn. [4.155].)

Let's rewrite these this way:

$$\left|\theta_\uparrow\right\rangle = \cos\left(\frac{\theta}{2}\right)\left|\uparrow\right\rangle + \sin\left(\frac{\theta}{2}\right)\left|\downarrow\right\rangle \qquad \left|\theta_\downarrow\right\rangle = \sin\left(\frac{\theta}{2}\right)\left|\uparrow\right\rangle - \cos\left(\frac{\theta}{2}\right)\left|\downarrow\right\rangle.$$

These formulas will come in handy for the problems on this homework set.

1. [5] The Einstein-Podolsky-Rosen "paradox" is based on the unusual properties of entangled states in quantum mechanics. One of the simplest examples is two particles in a spin singlet:

$$\left|\Psi_i\right\rangle = \left|s=0\right\rangle = \sqrt{\tfrac{1}{2}}\left(\left|\uparrow\right\rangle^{(1)}\left|\downarrow\right\rangle^{(2)} - \left|\downarrow\right\rangle^{(1)}\left|\uparrow\right\rangle^{(2)}\right)$$

Things get interesting if we send particle 1 to detector A, and particle 2 to detector B, where the two detectors are on opposite sides of the room. Each detector is a Stern-Gerlach apparatus whose axis can be set to any angle θ from the z-axis, in the x-z plane.

a) Show that $\left|\Psi_i\right\rangle$ has the same form regardless of what basis we choose. In other words, rewrite $\left|\Psi_i\right\rangle$ in terms of the eigenstates of $\vec{S}\cdot\hat{n}$, where $\hat{n}$ is a unit vector pointing at an arbitrary angle θ from the z-axis, in the x-z plane. The first step in your calculation should be to invert the basis transformation I wrote just above this problem. Your result shows that, no matter what direction the Stern-Gerlach detectors are pointed, the two detectors will always register opposite spin directions for the two particles <u>if the detectors are oriented along the same axis</u>.

b) The preceding result can be obtained in a more revealing way if we model the measurement process using projection operators. If detector A is oriented along the θ direction, and if we detect particle 1 as pointing "up" along that direction, then the state is projected onto:

$$\left|\Psi_f\right\rangle = \hat{P}^{(1)}(\theta_\uparrow)\left|\Psi_i\right\rangle = \left(\left|\theta_\uparrow\right\rangle^{(1)}\left\langle\theta_\uparrow\right|^{(1)} \otimes \hat{I}^{(2)}\right)\left|\Psi_i\right\rangle$$

Write $\left|\Psi_i\right\rangle$ in the S_z basis, as I did at the beginning of this problem, and plug it into this formula to find $\left|\Psi_f\right\rangle$. Then re-write $\left|\Psi_f\right\rangle$ in the $\vec{S}\cdot\hat{n}$ basis. It should be clear from your result what detector B must measure, if it is oriented along the same direction. (Note that $\left|\Psi_f\right\rangle$ is not normalized, because we have projected out only a part of the original state.)

(over)

2. [3] Griffiths problem 12.1.

3. [5] Griffiths problem 4.50. If you prefer, the relation to be shown can be re-written as:

$$\left\langle \left(\vec{S}^{(1)}\cdot\hat{a}\right)\left(\vec{S}^{(2)}\cdot\hat{b}\right)\right\rangle = -\frac{\hbar^2}{4}\hat{a}\cdot\hat{b}$$

where the expectation value is to be evaluated in the two-particle singlet state:

$$\left|\Psi(s=0)\right\rangle = \sqrt{\tfrac{1}{2}}\left(\left|\uparrow\right\rangle^{(1)}\left|\downarrow\right\rangle^{(2)} - \left|\downarrow\right\rangle^{(1)}\left|\uparrow\right\rangle^{(2)}\right).$$

Hint: Choose $\hat{a}$ along the z-axis, and $\hat{b}$ in the x-z plane at an angle θ from the z-axis.

4. [7] To test Bell's inequality in an EPR type experiment, the Stern-Gerlach detectors A and B must be allowed to be oriented in different directions. Let detector A, which will measure particle 1, be oriented at an angle θ from the z-axis, in the x-z plane. Detector B is oriented at an angle ϕ from the z-axis, also in the x-z plane.

a) Re-write $\left|\Psi(s=0)\right\rangle$ in terms of the basis states $\left|\theta_\uparrow\right\rangle^{(1)}$ and $\left|\theta_\downarrow\right\rangle^{(1)}$ for particle 1, and $\left|\phi_\uparrow\right\rangle^{(2)}$ and $\left|\phi_\downarrow\right\rangle^{(2)}$ for particle 2. Use the formulas you derived in part a) of problem 1.

b) If a detector measures the spin to be parallel to the detector orientation ("up" in that basis), we'll call that result +1. If the spin is "down" in that basis, we'll call that result -1. Evaluate the four probabilities: P(1,1), P(-1,-1), P(1,-1), and P(-1,1) with the two detectors oriented at angles θ and ϕ, respectively.

c) Calculate the average product of the two signals E(θ,ϕ) = P(1,1) + P(-1,-1) - P(1,-1) - P(-1,1). Your answer should be consistent with your answer to the previous problem (Griffiths 4.50).

d) Imagine that each detector can be oriented in two different directions, which can be different for the two detectors. For detector A, call them θ and θ', for detector B, call them ϕ and ϕ'. Define S = E(θ,ϕ) + E(θ',ϕ) + E(θ',ϕ') - E(θ,ϕ') Calculate S for θ = 0; θ'= π/2; ϕ = 0; ϕ' = π/2. Can an experiment with these detector orientations distinguish between the predictions of quantum mechanics and those of a "hidden-variable" theory? Recall that John Bell showed that a hidden variable theory will always produce the result -2 < S < 2.

Calculate S for θ = 0; θ'= π/2; ϕ = π/4; ϕ' = 3π/4. Answer the same question as above.