Physics 472 – Spring 2010

Homework #6, due Friday, February 26
(Point values are in parentheses.)

1. [7] Griffiths problem 6.7. For part (b), do not just use equation 6.27. Instead, write the 2 x 2 matrix representation of H' in the basis $|n\rangle$ and $|-n\rangle$, where $\langle x|n\rangle \equiv \Psi_n(x) = \frac{1}{\sqrt{L}} e^{2\pi i n x / L}$. In other words, evaluate the four elements of the matrix that Griffiths calls **W**:

$$\begin{pmatrix} \langle n|H'|n\rangle & \langle n|H'|-n\rangle \\ \langle -n|H'|n\rangle & \langle -n|H'|-n\rangle \end{pmatrix}$$

   After you evaluate these four numbers, find the eigenvalues and eigenvectors of the matrix. To simplify your calculation, I suggest you factor out the quantity $\dfrac{-aV_0\sqrt{\pi}}{L}$ from the matrix, and then call the off-diagonal terms δ_n. (You should obtain $\delta_n \propto e^{-(2\pi n a / L)^2}$.) You will find the following formula useful, which we derived in PHY471: $\int_{-\infty}^{\infty} e^{-\alpha x^2 + \beta x} dx = \sqrt{\dfrac{\pi}{\alpha}} e^{(\beta^2 / 4\alpha)}$. When you get to part (c), don't bother showing that using Eqn. (6.9) with the correct states gives you the correct energies. This is equivalent to showing that the eigenvalues you already calculated are just the diagonal elements of the diagonalized matrix! The answer to part (d) is Parity.

2. [6] Griffiths problem 6.9. To simplify your notation, label the three eigenvectors of H^0 as $|1\rangle$, $|2\rangle$, and $|3\rangle$. When you get to part (c), write down the 3 x 3 matrix form of H' in that basis. You can then read all the matrix elements you need directly from the matrix, without performing any matrix multiplication. When you get to part (d), don't be faked out when you discover that H' is already diagonal in the 2D subspace of degenerate states.

   Add a part (e) to the problem: Calculate the second-order shifts to states $|1\rangle$ and $|2\rangle$. You use the same formula [6.15] for second-order P.T., but now the sum is only over the states outside of the degenerate subspace, i.e. m=3 only. (That is because you have already exactly diagonalized H' within the 2D subspace.) With this last calculation, all three of your energies should agree with the expansion of the exact results to order ε^2.

3. [7] The isotropic 2-dimensional harmonic oscillator is easily solved by writing the Hamiltonian as a sum of x and y Hamiltonians:

$$\hat{H}^0 = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2m} + \tfrac{1}{2}m\omega^2(x^2 + y^2) = \hat{H}_x^0 + \hat{H}_y^0 \text{ with } [\hat{H}_x^0, \hat{H}_y^0] = 0.$$

   Simultaneous eigenstates of $\hat{H}_x^0$ and $\hat{H}_y^0$ obey $\hat{H}^0|n_x, n_y\rangle = (n_x + n_y + 1)\hbar\omega|n_x, n_y\rangle$.

a) Consider the perturbation $\lambda \hat{H}' = \lambda m\omega^2 xy$. Calculate the first and second order energy shifts of the ground state. In class we used $\hat{X} = \sqrt{\dfrac{\hbar}{2m\omega}}\left(\hat{a}_x^+ + \hat{a}_x\right)$ to evaluate the matrix elements:

$$\langle n_x'|x|n_x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n_x}\,\delta_{n_x',n_x-1} + \sqrt{n_x+1}\,\delta_{n_x',n_x+1}\right).$$ The same holds for $\hat{Y} = \sqrt{\dfrac{\hbar}{2m\omega}}\left(\hat{a}_y^+ + \hat{a}_y\right)$.

To evaluate the matrix elements of $\hat{H}'$ in the $|n_x n_y\rangle$ tensor product states, use:
$$\langle n_x'n_y'|xy|n_x n_y\rangle = \langle n_x'|x|n_x\rangle\langle n_y'|y|n_y\rangle$$

b) Use degenerate P.T. to calculate the first-order energy shifts of the first excited states, as well as the "correct" linear combinations of those two states that diagonalize $\hat{H}'$.

c) The full Hamiltonian, $\hat{H} = \hat{H}^0 + \lambda\hat{H}'$, is exactly solvable if you make the coordinate transformation $u = (x+y)/\sqrt{2}$, $v = (x-y)/\sqrt{2}$. Express $\hat{H}$ in terms of u, v, and their conjugate momenta P_u, and P_v. You should find that the harmonic oscillator in the "u" direction has a higher frequency than before, while in the "v" direction the frequency is lower. Calculate the exact energies of the new basis states $|n_u, n_v\rangle$. For the ground state, expand the energy to second order in λ. For the next two higher states, expand the energies to first order in λ. Compare your results with those you obtained in parts (a) and (b).