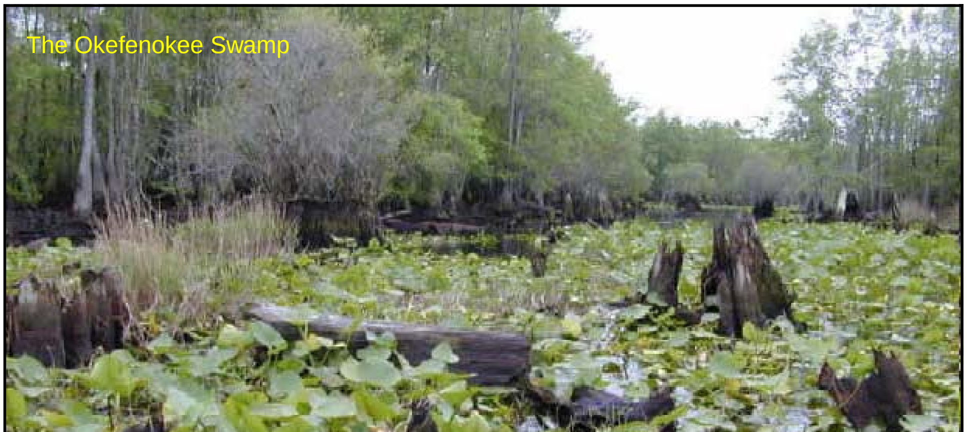
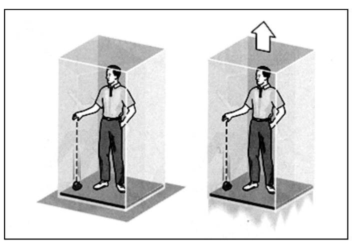


Some books about relativity:

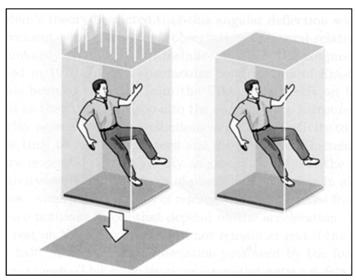
- Taylor & Wheeler *"Spacetime Physics"* = TW1
  (simple presentation, but deep insights)
- Taylor & Wheeler *"Introduction to Black Holes"* = TW2
  (CO-level math, deep insights)
- Weinberg *"Gravitation & Cosmology"*
  (hairy-looking math, but examples of solving field eqn)
- Hartle *"Gravity, An Introduction to Einstein's General Relativity"*
  (textbook for AST 860)

General Relativity (sort of)



Gravity

Upwards acceleration, no gravity.

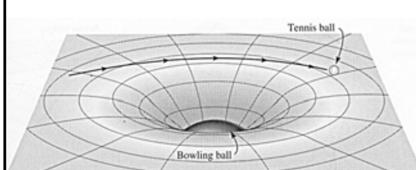


Falling due to gravity

No gravity

Equivalence Principle:

- Can't tell difference between gravity & acceleration
- …or between freefall & no gravity.
- So *any* experiment should give same answer in either case.



[CO 17.2]

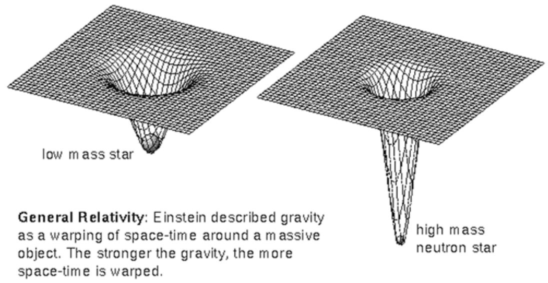
Gravity = Curved space-time

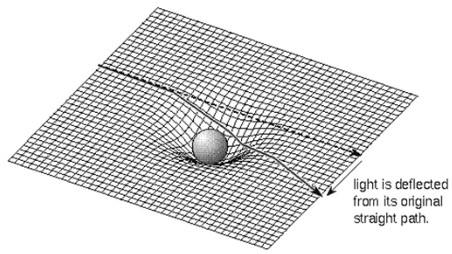
"Weak" equivalence principle:

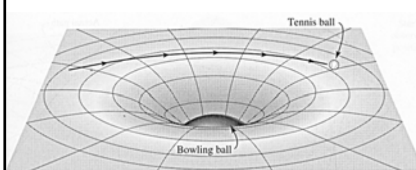
$$F = ma$$

$$F = \frac{GMm}{r^2}$$

What curves into where?



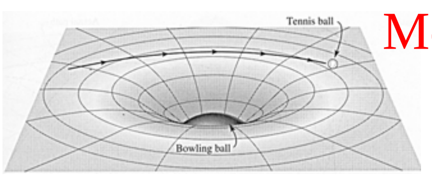


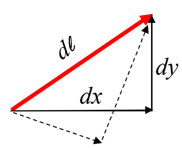


[CO 17.2]

Gravity = Curved space-time

Metrics





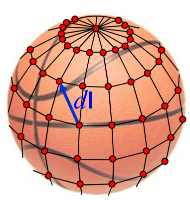
$d\ell^2 = dx^2 + dy^2$

$g_{xx} = g_{yy} = 1$

$g_{xy} = g_{yx} = 0$

$d\ell$ = invariant length

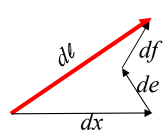
$$(d\ell)^2 = (rd\theta)^2 + (r\sin\theta\, d\phi)^2$$



Metric coefficients g_{ij}:

$$d\ell^2 = \sum g_{ij} dx_i dx_j$$

The set of all distances between grids of points, along all different coord. directions, specify shapes.



This coord. system needs a more complicated metric, with cross-terms.

Special Relativity

"interval"

$$(ds)^2 = (c\,dt)^2 - (d\ell)^2 = (c\,dt)^2 - \left[(dx)^2 + (dy)^2 + (dz)^2\right]$$

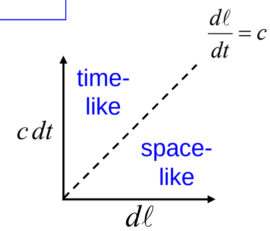
- Proper time:

  If $\Delta\ell = 0$: $\Delta\tau \equiv \frac{\Delta s}{c}$

  Proper distance:

  If $\Delta t = 0$: $\Delta\mathcal{L} = \sqrt{-(\Delta s)^2}$

  If $ds = 0$: $c^2 dt^2 = d\ell^2$; $\frac{d\ell}{dt} = c$ → light!



- Metric ←→ Lorentz transform

  [CO pg. 88; TW1 pg. 42]

$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}}$$

[CO (4.6, 4.9)]

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}}$$

Some Metrics

$$ds^2 = \sum g_{ij}\,dx_i\,dx_j$$

Flat space-time (special rel. = no gravity):

$$(ds)^2 = (c\,dt)^2 - (d\ell)^2 = (c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

Flat space-time, spherical coords:

$$(ds)^2 = (c\,dt)^2 - (dr)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

Space filled by uniform matter distribution:

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1 - k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

Empty space around point source of matter:

$$(ds)^2 = \left(c\,dt\sqrt{1 - 2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1 - 2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

Units, etc. -- to c or not to c:

$$d\tau^2 = \left[1 - \frac{2MG}{r}\right]dt^2 - \left[1 - \frac{2MG}{r}\right]^{-1} dr^2 - r^2\,d\theta - r^2\sin^2\theta\,d\varphi^2$$

$ct \to t$
(Weinberg)

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2$$

$ct \to t$; $GM/c^2 \to M$; leave out θ
(Taylor & Wheeler)

Vague outline of General Relativity

Stolen from Weinberg, "Gravitation & Cosmology"

CO Eqn. 17.15: $\mathcal{G} = -\frac{8\pi G}{c^4}\mathcal{T}$

Einstein's eqn: $\underbrace{R_{\mu\nu} - g_{\mu\nu}K}_{\text{Curvature}} = \underbrace{-8\pi G T_{\mu\nu}}_{\text{Mass-Energy}}$

Confusion Alarm!!!
The $R_{\mu\nu}$ used on this & following slide are *NOT* the Scale Factor!!

$T_{\mu\nu}$ = stress-energy tensor

At each point in space: $\begin{pmatrix} E & p_x & p_y & p_z \\ p_x & S_{xx} & S_{xy} & S_{xz} \\ p_y & S_{yx} & S_{yy} & S_{yz} \\ p_z & S_{zx} & S_{zy} & S_{zz} \end{pmatrix}$

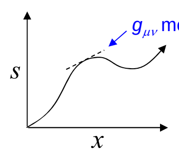
E = energy density
p = momentum flux
S = stress

K = Gaussian curvature
[CO eq. 29.103]

$g_{\mu\nu}$ = components of metric tensor

$R_{\mu\nu}$ = Ricci curvature tensor

Full of derivatives $dg_{\mu\nu}/dx$, etc.



Vague outline of General Relativity

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CO Eqn. 17.15: $\mathcal{G} = -\frac{8\pi G}{c^4}\mathcal{T}$

Einstein's eqn: $R_{\mu\nu} - g_{\mu\nu}K = -8\pi G T_{\mu\nu}$

Confusion Alarm!!!
The $R_{\mu\nu}$ used on this &

Same as:

$$\begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{pmatrix} - \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} K = -8\pi G \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{pmatrix}$$

Same as:

$$R_{11} - g_{11}K = -8\pi G T_{11}$$
$$R_{12} - g_{12}K = -8\pi G T_{12}$$
$$\vdots$$

(etc., for all 16 terms)

Cu

$T_{\mu\nu}$ = stress-energy tens

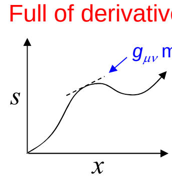
At e

K = Gaussian curvature
[CO eq. 29.103]

$g_{\mu\nu}$ = components of metric tensor

$R_{\mu\nu}$ = Ricci curvature tensor

Full of derivatives $dg_{\mu\nu}/dx$, etc.



Metric for flat space-time, spherical coords:

$$(ds)^2 = (dt)^2 - (dr)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

Stolen from Weinberg, "Gravitation & Cosmology"

Einstein's eqn: $R_{\mu\nu} - g_{\mu\nu}K = -8\pi G T_{\mu\nu}$

In empty space: $R_{\mu\nu} = 0$

Curved space-time: $ds^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)$

Unknown functions, allow space to be curved

Non-zero components of Einstein Eqn:

$$R_{rr} = \frac{B''(r)}{2B(r)} - \frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{A'(r)}{A(r)}\right) = 0$$

$$R_{\theta\theta} = -1 + \frac{r}{2A(r)}\left(-\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) + \frac{1}{A(r)} = 0$$

$$R_{\varphi\varphi} = \sin^2\theta R_{\theta\theta} = 0$$

$$R_{tt} = -\frac{B''(r)}{2A(r)} + \frac{1}{4}\left(\frac{B'(r)}{A(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{B'(r)}{A(r)}\right) = 0$$

Where:
$' = d/dr$
$'' = d^2/dr^2$

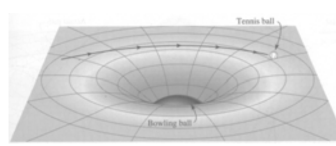
Schwarzschild's solution (1916):

$$ds^2 = \left[1 - \frac{2MG}{r}\right]dt^2 - \left[1 - \frac{2MG}{r}\right]^{-1}dr^2 - r^2\,d\theta - r^2\sin^2\theta\,d\varphi^2$$

Black Holes & the Schwarzschild Metric

Simplified metric, from Taylor & Wheeler2 (no c^2, no G, no θ):

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2$$



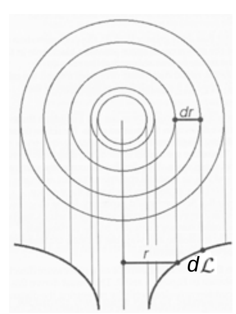
- Schwarzschild radius: $R_S = 2M$
- *Schwarzschild coordinates*
  - reconstructed as if seen from a point where space is flat.
- Metric for observer sitting on a shell at r

Proper time:

$$d\tau = dt_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} dt$$

Proper distance:

$$d\mathcal{L} = dr_{\text{shell}} = \frac{dr}{\left(1 - \frac{2M}{r}\right)^{1/2}}$$

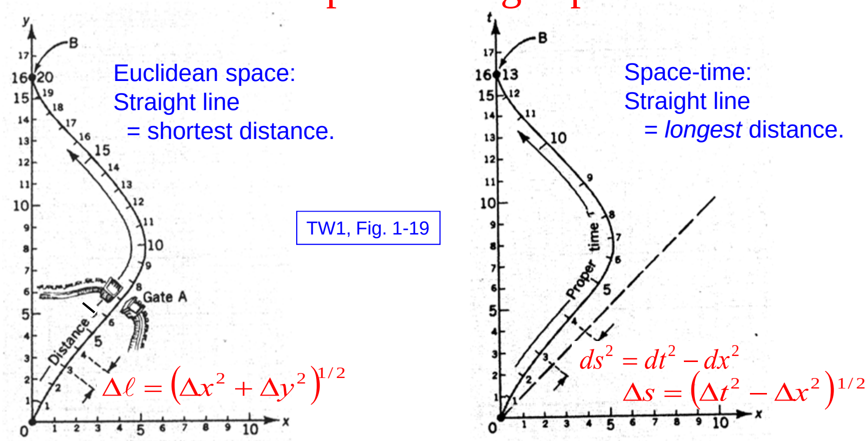


➔ Observer measures things in locally flat space-time:

$$ds^2 = dt^2_{\text{shell}} - dr^2_{\text{shell}} - r^2 d\phi^2$$

- Free-falling astronaut:
  - Metric = flat space-time in local region.
  - Time on wristwatch = $\tau = s$.

World-Lines = path through space-time



TW1, Fig. 1-19

Geodesic = straightest possible world line

- In free-fall, objects follow geodesics = extremum of $\int ds$ $\quad (= \int d\tau)$
  - Normally a maximum
    ➔ *observer travelling on Geodesic sees max. time pass*
- Light always follows null geodesic, with $\Delta s = \Delta\tau = 0$

$$\frac{ds}{d\,?} = 0$$

Geodesics

Example in CO (pgs. 632-633)

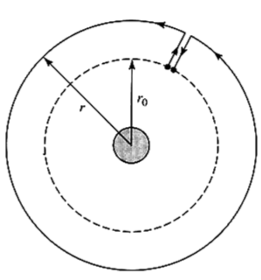
$$(ds)^2 = \left(c\,dt\sqrt{1 - 2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1 - 2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

A satellite in circular orbit.

Assume:
- Circular orbit ➔ $dr = 0$
- In plane where $d\phi = 0$
- Moving at "specified angular speed" ω ➔ $d\theta = \omega\,dt$

$$(ds)^2 = \left[\left(c\sqrt{1 - 2GM/rc^2}\right)^2 - r^2\omega^2\right]dt^2 = \left(c^2 - \frac{2GM}{r} - r^2\omega^2\right)dt^2$$

$$\Delta s = \int_0^{2\pi/\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\,dt.$$



Find r that makes Δs be an extremum:

$$\frac{d}{dr}(\Delta s) = \frac{d}{dr}\left(\int_0^{2\pi/\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\,dt\right) = 0$$

$$\frac{d}{dr}\sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2} = 0$$

$$\frac{2GM}{r^2} - 2r\omega^2 = 0$$

$$v = r\omega = \sqrt{\frac{GM}{r}}$$

A familiar Newtonian result!

Metric for flat space-time, spherical coords:

Stolen from Weinberg, "Gravitation & Cosmology"

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Curved Spaces & the Robertson-Walker Metric

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

[CO] notation, with c^2

- R-W metric: most general solution for universe obeying Cosmological Principle.
  - Homogeneous & Isotropic
  - Smooth distribution of matter.
  - Same everywhere at any given time.

- The non-zero components of the Einstein equation then reduce to

Back to Weinberg's notation, without c^2

$\dot{R} = \frac{dR}{dt}$ etc.

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2$$

← Friedmann eqn.[29.10]

$$\dot{p}R^3 = \frac{d}{dt}\{R^3[\rho + p]\}$$

$$\frac{d}{dR}(\rho R^3) = -3pR^2$$

← Form of fluid eqn.[29.50]

Stolen from Weinberg, "Gravitation & Cosmology"