Curved Spaces & the Robertson-Walker Metric

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\, d\theta)^2 + (\varpi \sin\theta\, d\phi)^2\right]$$

[CO] notation, with c^2

- R-W metric: most general solution for universe obeying Cosmological Principle.
  - Homogeneous & Isotropic
  - Smooth distribution of matter.
  - Same everywhere at any given time.

- The non-zero components of the Einstein equation then reduce to

Back to Weinberg's notation, without c^2

$\dot{R} = \frac{dR}{dt}$ etc.

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2$$

Friedmann eqn.[29.10]

$$\dot{p}R^3 = \frac{d}{dt}\{R^3[\rho + p]\}$$

$$\frac{d}{dR}(\rho R^3) = -3pR^2$$

Form of fluid eqn.[29.50]

Stolen from Weinberg, "Gravitation & Cosmology"

Curved Spaces & the Robertson-Walker Metric

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\, d\theta)^2 + (\varpi \sin\theta\, d\phi)^2\right]$$

[CO] notation, with c^2

- R-W metric: most general solution for universe obeying Cosmological Principle.
  - Homogeneous & Isotropic
  - Smooth distribution of matter.
  - Same everywhere at any given time.
- Curvature

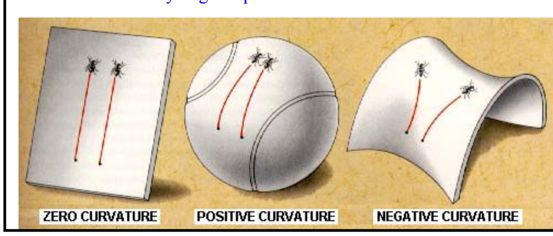
[CO sect. 29.3]

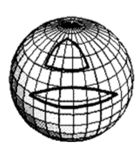
$$K \equiv \frac{k}{R^2(t)}$$

$$k \equiv \frac{1}{\Re_0^2}$$

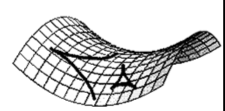
$\Re_0$ = Present radius of curvature (meters)

  - Can be found from local measurements
    - By bug on sphere

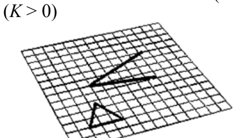




Positive Curvature ($K > 0$)



Negative Curvature ($K < 0$)



Flat ($K = 0$)

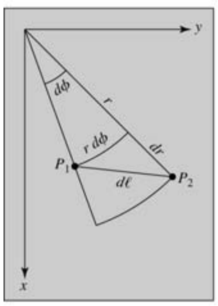
Geometry of a 2D Spherical Surface

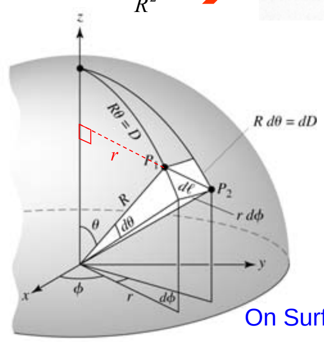
$$r = R\sin\theta$$
$$dr = R\cos\theta\, d\theta$$

$$R\,d\theta = \frac{dr}{\cos\theta} = \frac{R\,dr}{\sqrt{R^2 - r^2}} = \frac{dr}{\sqrt{1 - r^2/R^2}} \;\rightarrow\; \begin{aligned}(d\ell)^2 &= (dD)^2 + (r\,d\phi)^2 = (R\,d\theta)^2 + (r\,d\phi)^2 \\ (d\ell)^2 &= \left(\frac{dr}{\sqrt{1 - r^2/R^2}}\right)^2 + (r\,d\phi)^2\end{aligned}$$

$$K = \frac{1}{R^2} \;\rightarrow\; (d\ell)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r\,d\phi)^2$$

On Flat Surface





On Surface of Ball

Geometry of a 2D Spherical Surface

$$r = R\sin\theta$$
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$$R\,d\theta = \frac{dr}{\cos\theta} = \frac{R\,dr}{\sqrt{R^2 - r^2}} = \frac{dr}{\sqrt{1 - r^2/R^2}} \;\rightarrow\; \begin{aligned}(d\ell)^2 &= (dD)^2 + (r\,d\phi)^2 = (R\,d\theta)^2 + (r\,d\phi)^2 \\ (d\ell)^2 &= \left(\frac{dr}{\sqrt{1 - r^2/R^2}}\right)^2 + (r\,d\phi)^2\end{aligned}$$

$$K = \frac{1}{R^2} \;\rightarrow\; (d\ell)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r\,d\phi)^2$$

To get R-W metric:

- Add another dimension

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2$$

- Add time

$$(ds)^2 = (c\,dt)^2 - \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1 - k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

where $r(t) = R(t)\varpi$ and $K(t) \equiv \dfrac{k}{R^2(t)}$

Geometry of a 3D Surface (our Universe) in an Expanding 4D Space

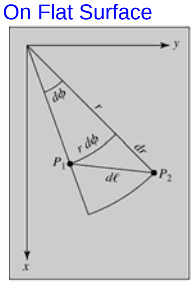
$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\, d\theta)^2 + (\varpi \sin\theta\, d\phi)^2\right]$$

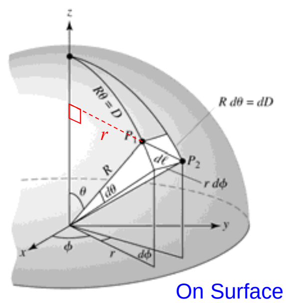
$$r(t) = R(t)\varpi$$

$$K(t) \equiv \frac{k}{R^2(t)}$$

Area of sphere at co-moving distance ϖ: $A = 4\pi\varpi^2$

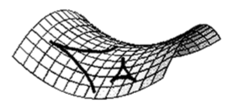
On Flat Surface



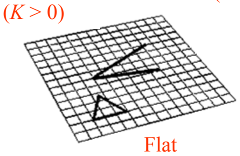


On Surface of Ball

Positive Curvature ($K > 0$)



Negative Curvature ($K < 0$)



Flat ($K = 0$)

Dynamics of a 3D Surface (our Universe) in an Expanding 4D Space

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\, d\theta)^2 + (\varpi \sin\theta\, d\phi)^2\right]$$

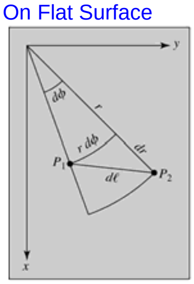
$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2$$

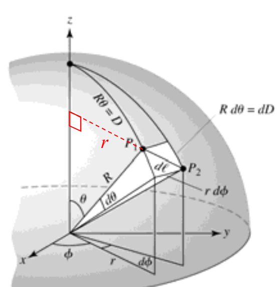
Friedmann Equation

$$r(t) = R(t)\varpi$$

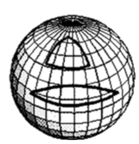
$$K(t) \equiv \frac{k}{R^2(t)}$$

On Flat Surface

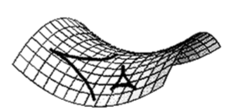




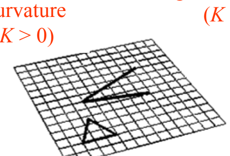
On Surface of Ball



Positive Curvature ($K > 0$)



Negative Curvature ($K < 0$)



Flat ($K = 0$)

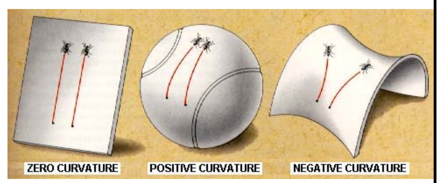
Dynamics of a 3D Surface (our Universe) in an Expanding 4D Space

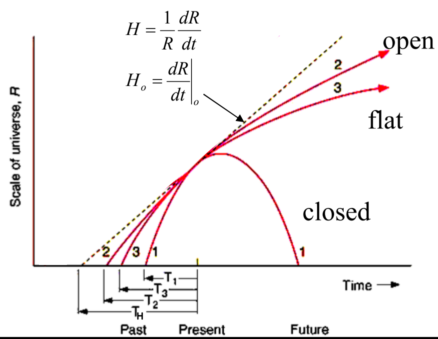
$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\, d\theta)^2 + (\varpi \sin\theta\, d\phi)^2\right]$$

$$r(t) = R(t)\varpi$$

$$K(t) \equiv \frac{k}{R^2(t)}$$

$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2$$

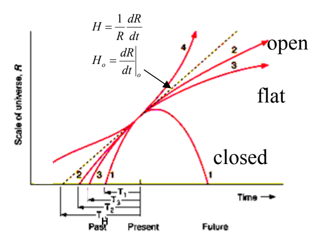




- Dynamics and curvature both due to mass-energy density.
- For Friedmann Eqn *without* Dark Energy:

Density	Curvature	Dynamics
$\rho_o < \rho_{c,o}$	Negative	Expands forever
$\rho_o = \rho_{c,o}$	Flat	Oozes to stop at $t = \infty$
$\rho_o > \rho_{c,o}$	Positive	Collapses back

Sneak Preview



Accelerating Universe

- Hubble's law:

  $v(t) = H(t)\, r(t)$

- Lookback time ➔ for more distant objects, we measure $H(t)$ at earlier t.
- If gravity constantly slows expansion, expect larger H at earlier t.
- Late 1990s: Type Ia SNe showed $H(t)$ is currently increasing with time.

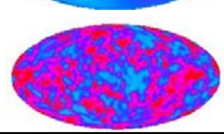
Flat Universe

- CMB fluctuations
- 1 part in 10^5 amplitude
- COBE: angular resolution too low
- WMAP has higher angular resolution.
- Fluctuations have known physical size.
- Angular size depends on geometry of universe.

COBE results

Blue = 0ºK
Red = 4ºK

Blue = 2.724ºK
Red = 2.732ºK
Dipole Anistropy
➔motion of Sun through Universe.



After removing dipole
Red – blue = 0.0002ºK

The Cosmological Constant

Einstein's Eqn: $$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

One last possible term. Λ = arbitrary constant.

The "complete" Friedmann Eqn: $$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right] R^2 = -kc^2$$

Λ acts as outward force:

Newtonian version: $$\frac{1}{2}mv^2 - G\frac{M_r m}{r} - \frac{1}{6}\Lambda m c^2 r^2 = -\frac{1}{2} m k c^2 \varpi^2$$

Define a potential:

$$U_\Lambda \equiv -\frac{1}{6}\Lambda m c^2 r^2 \rightarrow \mathbf{F}_\Lambda = -\frac{\partial U_\Lambda}{\partial r}\hat{\mathbf{r}} = \frac{1}{3}\Lambda m c^2 r\,\hat{\mathbf{r}} \quad [29.109]$$

The acceleration eqn: $$\frac{d^2R}{dt^2} = \left\{-\frac{4}{3}\pi G\left[\rho_m + \rho_{\rm rel} + \frac{3(P_m + P_{\rm rel})}{c^2}\right] + \frac{1}{3}\Lambda c^2\right\} R \quad [29.112]$$

Psst... is it a constant?

Equation of state [CO pp. 1161-1162]

- Relation between *P*, *R* and ρ

Define: $\rho = \rho_o R^{-3(1+w)}$

Fluid eqn (29.50): $\frac{d(R^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(R^3)}{dt}$

[29.52] $P = wu = w\rho c^2$

$\rightarrow$

Matter: $w = 0$ $\quad \rho \propto R^{-3}$

Radiation: $w = 1/3$ $\quad \rho \propto R^{-4}$

Cosm. Const: $w = -1$ $\quad \rho \propto R^{0}$

$P \propto$ energy density of fluid

Cosmological Constant as a "Negative Pressure"

Friedmann Eq. with Cosmological Constant: $$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right] R^2 = -kc^2$$

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} = \text{constant} = \rho_{\Lambda,0} \rightarrow \left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G(\rho_m + \rho_{\rm rel} + \rho_\Lambda)\right] R^2 = -kc^2 \quad [29.114]$$

Acceleration Eqn:

$$[29.111] \quad \frac{d(R^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(R^3)}{dt} \rightarrow \frac{d^2R}{dt^2} = \left\{-\frac{4}{3}\pi G\left[\rho_m + \rho_{\rm rel} + \frac{3(P_m + P_{\rm rel})}{c^2}\right] + \frac{1}{3}\Lambda c^2\right\} R$$

Negative pressure (29.115)

$$P_\Lambda = -\rho_\Lambda c^2 \rightarrow \frac{d^2R}{dt^2} = \left\{-\frac{4}{3}\pi G\left[\rho_m + \rho_{\rm rel} + \rho_\Lambda + \frac{3(\overset{0}{\cancel{P_m}} + \overset{\rho_{rel}c^2/3}{\cancel{P_{\rm rel}}} + \overset{-\rho_\Lambda c^2}{\cancel{P_\Lambda}})}{c^2}\right]\right\} R$$

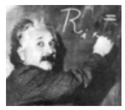
$$= -\frac{4}{3}\pi G\left[\rho_m + 2\rho_{rel} - 2\rho_\Lambda\right] R = -\frac{H^2}{2\rho_c}\left[\rho_m + 2\rho_{rel} - 2\rho_\Lambda\right] R$$

Psst... is it a constant? Is $w = -1$?

using $\frac{8\pi G\rho_c}{3} = H^2$

A slight renaming….

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right]R^2 = -kc^2$$


Cosmological Constant

A slight renaming….

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right]R^2 = -kc^2$$ [29.108]

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} = \text{constant} = \rho_{\Lambda,0}$$ [29.113]

$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\left(\rho_m + \rho_{rel} + \rho_\Lambda\right)\right)R^2 = -kc^2$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda c^2}{3H^2}$$

[29.9]

$$\Omega \equiv \Omega_m + \Omega_{\text{rel}} + \Omega_\Lambda$$ [29.119]

$$\left(H^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2 \quad \Rightarrow \quad H^2(1-\Omega)R^2 = -kc^2$$ Friedmann Eqn

$$H_0^2(1-\Omega_0) = -kc^2$$ [29.121]

$$H_0^2(1-\Omega_0) = -kc^2$$ [29.121]

The basic WMAP result: $k = 0$

$$\Rightarrow \quad [\Omega_0]_{\mathrm{WMAP}} = 1.02 \pm 0.02$$

$$\Omega_0 = \Omega_{m,0} + \Omega_{\mathrm{rel},0} + \Omega_{\Lambda,0} = 1$$

$$[\Omega_{m,0}]_{\mathrm{WMAP}} = 0.27 \pm 0.04,$$

$$\Omega_{\mathrm{rel},0} = 8.24 \times 10^{-5}$$

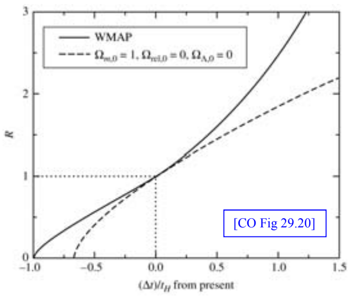
$$[\Omega_{\Lambda,0}]_{\mathrm{WMAP}} = 0.73 \pm 0.04,$$

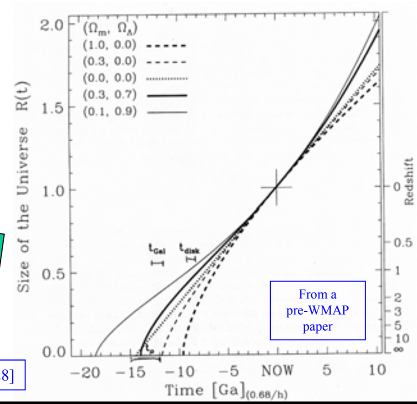
Friedmann Eqn:

$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G(\rho_m + \rho_{rel} + \rho_\Lambda)\right)R^2 = -kc^2$$

Solution for $k = 0$:

$$t = \sqrt{\frac{3}{8\pi G}} \int_0^R \frac{R'\,dR'}{\sqrt{\rho_{m,0}R' + \rho_{\mathrm{rel},0} + \rho_{\Lambda,0}R'^4}}.$$ [29.128]





Some Universes

Open *vs.* Closed:

$$\left(H^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2$$

$$H^2(1-\Omega)R^2 = -kc^2$$

$$H_0^2(1-\Omega_0) = -kc^2$$

$$k = 0 \Rightarrow \quad \Omega_0 = \Omega_{m,0} + \underbrace{\cancel{\Omega_{\mathrm{rel},0}}}_{\sim 0} + \Omega_{\Lambda,0} = 1$$

Accelerating *vs.* Decelerating:

Deceleration parameter:
$$q(t) = -\frac{R(t)\left[d^2R(t)/dt^2\right]}{\left[dR(t)/dt\right]^2}$$

+ accel. eqn:

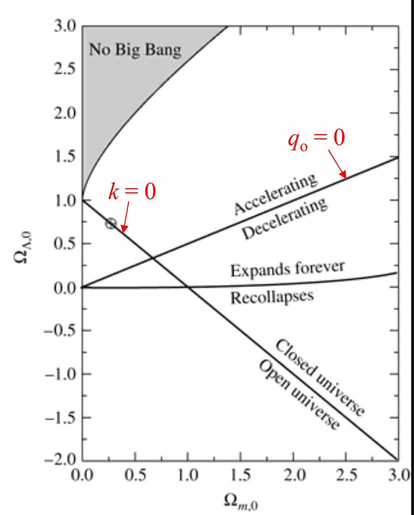
$$\frac{d^2R}{dt^2} = -\frac{H^2}{2\rho_c}\left[\rho_m + 2\rho_{rel} - 2\rho_\Lambda\right] R$$

(Camouflaged accel. eqn.)

$$q(t) = \frac{1}{2}\Omega_m(t) + \underbrace{\cancel{\Omega_{\mathrm{rel}}(t)}}_{\sim 0} - \Omega_\Lambda(t)$$

Expands Forever *vs.* Recollapses:

Does dR/dt ever = 0? See [29.135]



[29.4] Observational Cosmology

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

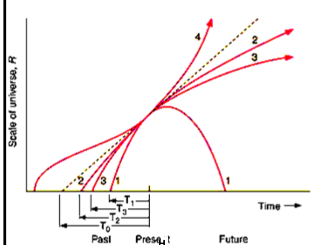
$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right]R^2 = -kc^2$$

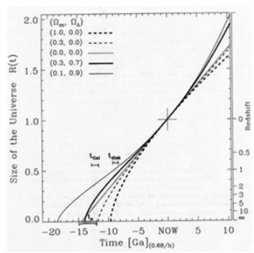
$$H_0^2(1-\Omega_0) = -kc^2$$

- Some theoretical parameter sets:
  - *R(t)* vs. *t*
  - $\Omega_{\Lambda,o}$ vs. $\Omega_{m,o}$
  - Curvature *k, dR/dt,* d^2R/dt^2

- But what can we actually *measure* that will tell us which universe we live in?

  As a function of *z*:
  - Apparent mag. of standard candles.
  - Angular sizes.
  - Space density of galaxies.





Are two numbers enough?