AST 308
Homework Assignment 7
Due Tuesday Dec 6

This homework goes with the lecture material about Black Holes that I plan to present on Thursday Dec. 1. The idea is for you to combine the equation for the conservation of energy in a Schwarzschild geometry with the appropriate metric to derive a few equations of motion for objects moving in a gravitational field. In case you are interested, the equation for conservation of energy is derived on the last page of this assignment. It is easier than it looks.

Questions 1 and 2. The astronaut starts out at rest at $r = \infty$ (*i.e.* he/she is just gradually pulled into the black hole and accelerated by gravity; nobody is given a shove). The astronaut has no angular momentum, so falls straight into the black hole.

Astronaut Falling into a Black Hole

- Schw. conservation of energy: $\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{ds}$
- Schw. metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2$$

This assignment uses the metric with the constants G and $c = 1$, and leaving out the θ coordinate (i.e. assuming motion in a plane)

Eq. of motion for freefall in Schw. Coords:

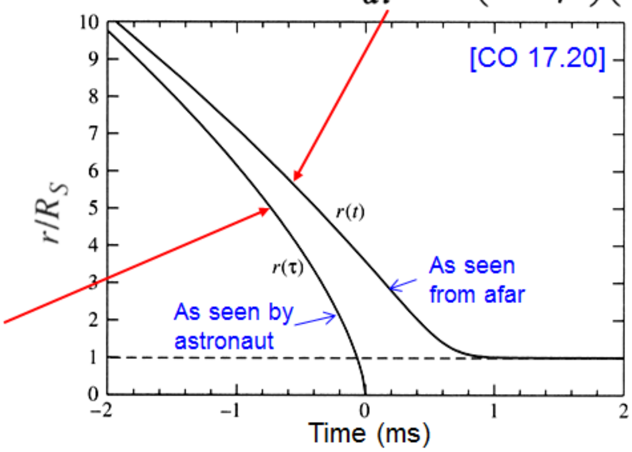
Homework Q. 1: Show this →

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{1/2}$$

Homework Q. 2: Show this

For free-falling astronaut, time steps on wristwatch = *ds*, so velocity is:

$$\frac{dr}{ds} = -\left(\frac{2M}{r}\right)^{1/2}$$



Hints for Question 1:

- What is the rest energy/per unit mass for an object outside of the gravitational field (i.e. at $r = \infty$)?
- Since energy is conserved, what is E/m always equal to?

In the [CO] notation, $E = mc^2$
But in the notation used here $c = 1$,
so $E/m = 1$ at $r = \infty$

$E/m = 1$ at any r, so we have

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{ds} = 1$$

- What is the value of $d\phi$ in this problem?
- The rest is algebra.

Hint for Question 2: $\frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds}$ and you know both $\frac{dr}{dt}$ and $\frac{dt}{ds}$ from Question 1.

Question 3:

Given the Schwarzschild metric, energy and angular momentum definitions shown at the top of the slide, show that the equation for the effective potential in the Schwarzschild geometry is correct:

Orbits around black holes

(from TW2)

- Schw. metric $ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2$
- Schw. conservation of energy: $\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{ds}$
- Schw. conservation of angular mom: $r^2 \frac{d\phi}{ds}$ = constant $\frac{L}{m} = r^2 \frac{d\phi}{ds}$
- Effective potential
  - Newtonian

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{E}{m} - \left[-\frac{M}{r} + \frac{(L/m)^2}{2r^2}\right]$$

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{E}{m} - \frac{V(r)}{m}$$

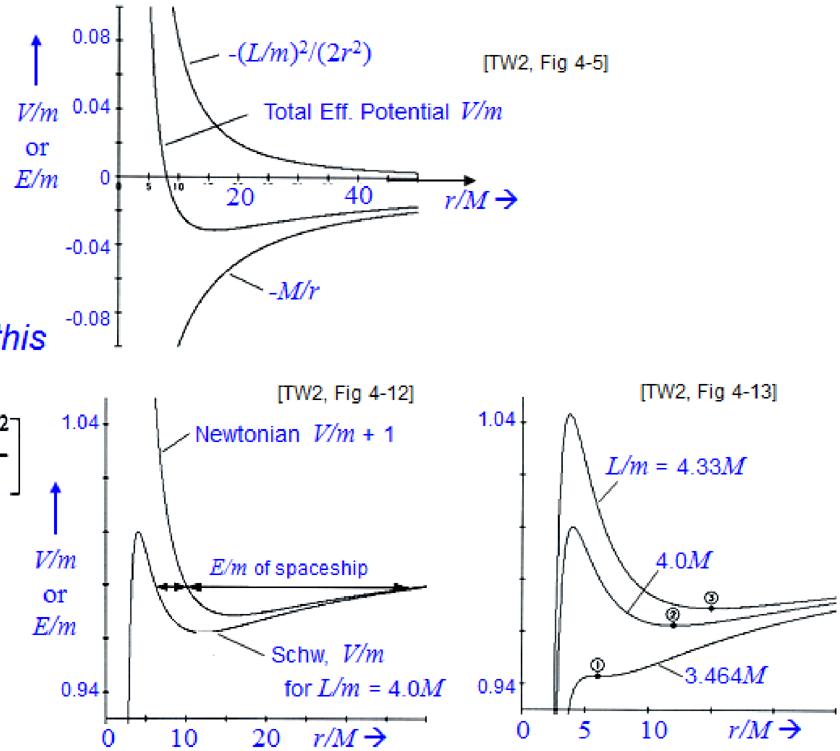
Homework Q. 3: Show this

  - Schwarzschild

$$\left(\frac{dr}{ds}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left[1 + \frac{(L/m)^2}{r^2}\right]$$

$$\left(\frac{dr}{ds}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(\frac{V}{m}\right)^2$$

- Precession of perihelion
- Innermost stable orbit



Hint:

Once you have the metric and the definitions of energy and angular momentum (all of which are given in this problem), then it's just algebra. But the algebra leads you to interesting results.

Appendix:
Deriving the Energy per Unit Mass for the Schwarzschild Geometry:

Use the idea that objects follow *geodesics* = paths through space-time over which $\int ds$ is an extremum.

Suppose that an object with no angular momentum (so that $d\phi = 0$) is falling in a gravitational field. Points r_1, r_2 and r_3 all lie along its path, but to know the object's motion through space-time we need to work out the times at which it will pass through those points.

Schwazschild metric: $$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\phi^2$$ ($d\phi^2 \rightarrow 0$)

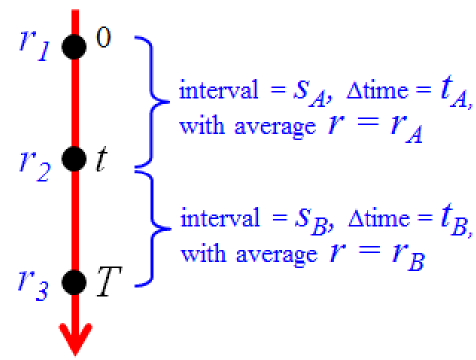
Rewrite as:

(where s is now some arbitrary segment of a longer path [or *interval*] through space-time)

$$\Delta s^2 = \left(1-\frac{2M}{r}\right)\Delta t^2 + const$$

$$s^2 = \left(1-\frac{2M}{r}\right)t^2 + const$$

Here, *const* just means terms not involving t



Break s into two smaller intervals s_A and s_B. Require the falling object to pass through point $r = r_1$ at time 0 and through $r = r_3$ at time T. At what intermediate time t will it pass through the fixed point r_2? G.R. says that object will follow a geodesic, so that t automatically adusts itself so that s is an extremum. i.e. $ds/dt = 0$.

The clever trick: Find ds/dt separately for each of the two segments s_A and s_B:

$$s_A = \left[\left(1-\frac{2M}{r_A}\right)t^2 + const\right]^{1/2}$$

$$\frac{ds_A}{dt} = \frac{1}{2}\left[\left(1-\frac{2M}{r_A}\right)t^2 + const\right]^{-1/2}\left(1-\frac{2M}{r_A}\right)2t$$

$$\frac{ds_A}{dt} = \left(1-\frac{2M}{r_A}\right)\frac{t_A}{s_A}$$

$$s_B = \left[\left(1-\frac{2M}{r_B}\right)(T-t)^2 + const\right]^{1/2}$$

$$\frac{ds_B}{dt} = -\frac{1}{2}\left[\left(1-\frac{2M}{r_B}\right)(T-t)^2 + const\right]^{-1/2}\left(1-\frac{2M}{r_B}\right)2(T-t)$$

$$\frac{ds_B}{dt} = -\left(1-\frac{2M}{r_B}\right)\frac{(T-t)}{s_B} = -\left(1-\frac{2M}{r_B}\right)\frac{t_B}{s_B}$$

Condition for the extremum: $$\frac{ds_A}{dt} + \frac{ds_B}{dt} = \frac{ds}{dt} = 0$$

$$\left(1-\frac{2M}{r_A}\right)\frac{t_A}{s_A} = \left(1-\frac{2M}{r_B}\right)\frac{t_B}{s_B} = \left(1-\frac{2M}{r}\right)\frac{\Delta t}{\Delta s} = \left(1-\frac{2M}{r}\right)\frac{dt}{ds} = \frac{E}{m}$$

A constant of motion, for any $\Delta t/\Delta s$

Let's arbitrarily call it "Energy per unit mass"

The above derivation is from Taylor & Wheeler, *"Exploring Black Holes, Introduction to General Relativity"* (TW2). They show a similar derivation of the energy equation for special relativity, and also many very well-explained examples of using the Schwarzschild metric to determine how things move through a gravitational field. Highly recommended