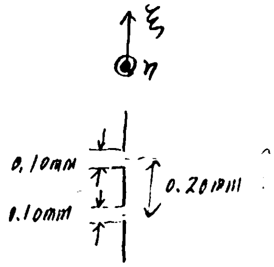
1)



$\xi_{max} = 0.15mm$

check

$$\frac{k\,\xi_{max}^2}{2} = \frac{\pi\,\xi_{max}^2}{\lambda} = \frac{\pi \times (0.15\times10^{-3}\,m)^2}{5\times10^{-9}\,m} \approx 0.14\,m \ll 2.5\,m$$

$\therefore$ Fraunhofer approximation is valid.

Note: Of course Fresnel approximation is valid, too.

For Fraunhofer region, the 'far-field' electric field amplitude

$$U(x,y) = \frac{e^{-ikz}\,e^{-i\frac{k}{2z}(x^2+y^2)}}{-i\lambda z}\iint d\xi\, d\eta\, U(\xi,\eta)\, e^{+i(k_x\xi + k_y\eta)}$$

$$k_x = \frac{2\pi}{\lambda}\sin\theta = \frac{2\pi}{\lambda}\times\frac{x}{z} = 2\pi f_x \qquad f_x \equiv \frac{x}{\lambda z}$$

For long slit oriented in η-direction, we only to labor in integrating in ξ. [$U^*(x,y)$ in y-direction (parallel to η-axis) is δ-function or zero actually. $\because \int_{-\infty}^{+\infty} d\eta\, e^{-i2\pi f_y \eta} = 2\pi\,\delta(2\pi f_y)$]

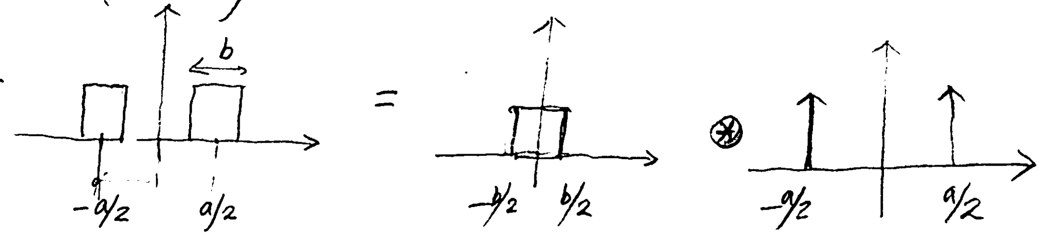
$$U(x) \propto \int d\xi\, U(\xi)\, e^{i2\pi f_x \xi} = \mathcal{F}\{U(\xi)\}$$

We drop out the phase and weighting term just for simplicity.

$f \otimes g$ convolution

Note $\mathcal{F}\{f\otimes g\} = \mathcal{F}\{f\}\times\mathcal{F}\{g\}$

$U(\xi) =$



$\mathrm{rect}(\frac{\xi}{b})$ $\qquad \delta(\xi+\frac{a}{2})+\delta(\xi-\frac{a}{2})$

$$\mathcal{F}\{U(\xi)\} = \mathcal{F}\{\mathrm{rect}(\tfrac{\xi}{b})\} \times \mathcal{F}\{\delta(\xi+a/2)+\delta(\xi-a/2)\}$$

$$= \left(b\,\frac{\sin\pi b f_x}{\pi b f_x}\right)\times\left(\cos\pi a f_x\right)$$

$$U(x) \propto \frac{\sin\pi b f_x}{\pi b f_x}\times\cos\pi a f_x$$

$$I(x) \propto |U(x)|^2 = \left(\frac{\sin\pi b f_x}{\pi b f_x}\right)^2\cdot\cos^2\pi a f_x$$

There are two components in $I(x)$

(1) $\left(\dfrac{\sin \pi b f_x}{\pi b f_x}\right)^2 = \mathrm{sinc}^2(\pi b f_x)$: due to slit width

(2) $(\cos \pi a f_x)^2$ ——— > due to slit seperation

First zero of (1)

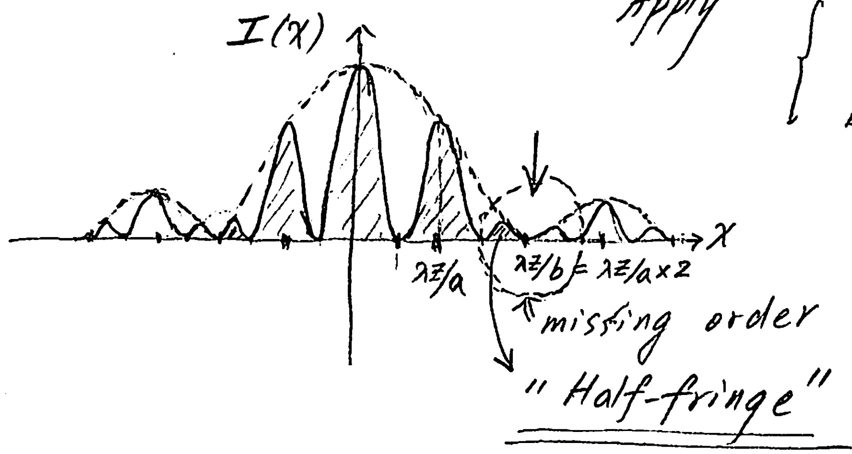
$\Rightarrow \quad \pi b f_x = \dfrac{\pi b x_0}{\lambda z} = \pi \;\Rightarrow\; x_0 = \dfrac{\lambda z}{b}$

Peaks of (2)

$\pi a f_x = \dfrac{\pi a x_a}{\lambda z} = m\pi \;\Rightarrow\; x_a = \dfrac{\lambda z}{a} \times m$

$m \in 0, 1, 2, \dots$

Apply $\begin{cases} a = \dfrac{0.20}{2} = 0.10\ mm \\ b = \dfrac{0.10}{2} = 0.05\ mm \end{cases}$



A scatch of the diffraction pattern is shown for $a/b = 2$. The missing order occurs at $\lambda z/b = 2 \cdot \dfrac{\lambda z}{a}$, and the fringe next to it is counted as "half".

There are four fringes inside the first bright zone. (3 + 2 × "half-fringe" = 4 fringes)

It's O.K. if you say there are three or five fringes as long as you state clearly what you mean.

2) For double-slit, the intensity of diffraction pattern

$$I \propto \text{sinc}^2\left(\frac{\pi b x}{\lambda z}\right) \cos^2\left(\frac{\pi d x}{\lambda z}\right) = \text{sinc}^2(\alpha) \cos^2(\beta)$$

where b is the width of each slit, d the seperation of their centers, z the distance between the screen and the double slit, and λ the wavelength of the incident light.

For fringes, the minima are given by $\beta = (n+\frac{1}{2})\pi$, $n = \pm 1, \pm 2 \ldots$

Thus the fringe spacing Δx is given by

$$\pi \frac{\Delta x \cdot d}{\lambda z} = \pi \rightarrow \Delta x = \frac{\lambda z}{d}$$

Hence, for $\Delta y = 1\text{cm}$ as from the figure,

$$d = \lambda D / \Delta x = 6 \times 10^{-3}\ \text{cm} = 60\ \mu m.$$

The pattern shows the missing order $d/b = 4$, from which we get the width of each slit

$$b = d/4 = 1.5 \times 10^{-3}\ \text{cm} = 15\ \mu m.$$

2. (A)

$$\Delta d = N \frac{\lambda_0}{2} = 1000 \times 5 \times 10^{-7}\ m / 2 = 2.50 \times 10^{-4}\ m.$$

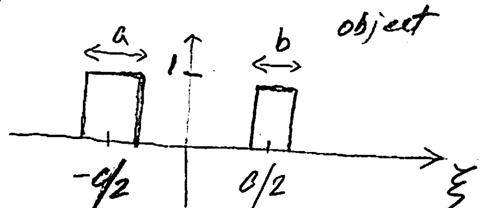
(B) optical path difference

$$\Lambda = \Delta d = N(\lambda_0/2) \ ; \ \Lambda = (n_{air} \cdot x - n_{vacuum}\ x)\ ;$$

$$N = \frac{2\Lambda}{\lambda_0} = 2[(1.00029 - 1.0000) \times 0.10\ m] / 6 \times 10^{-7}\ m$$

$$= 97$$

3)



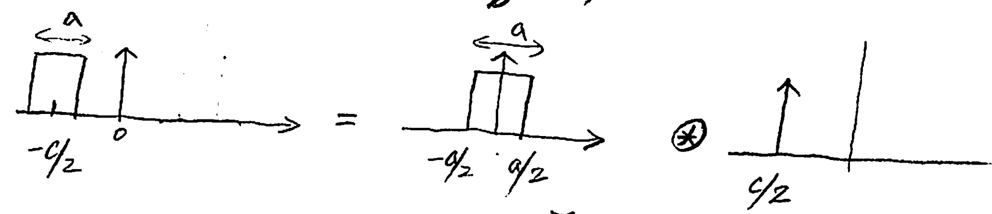
The diffraction pattern is

$$\propto \int_{-c/2-a/2}^{-c/2+a/2} e^{ik_x\xi}\, d\xi + \int_{+c/2-b/2}^{+c/2+b/2} e^{ik_x\xi}\, d\xi$$

$$k_x = \frac{2\pi}{\lambda}\cdot \sin\theta = \frac{2\pi}{\lambda}\cdot\frac{x}{z} \quad \#$$

$$U(\xi) = \text{rect}\left(\frac{\xi + c/2}{a}\right) + \text{rect}\left(\frac{\xi - c/2}{b}\right)$$

$$\text{rect}\left(\frac{x+c/2}{a}\right) =$$



$\text{rect}\left(\frac{\xi}{a}\right)$ $\quad$ $\delta(\xi + c/2)$

$$\mathcal{F}\left\{\text{rect}\left(\frac{x+c/2}{a}\right)\right\} = \mathcal{F}\{\text{rect}(\xi/a)\} \times \mathcal{F}\{\delta(\xi + c/2)\}$$

$$= a\ \text{sinc}(\pi a f_x) \times e^{+i2\pi f_x(-c/2)}$$

$$= e^{+i\pi f_x \cdot c} \cdot a\ \text{sinc}(\pi a f_x) \equiv U_a$$

Similarly for slit b, $\quad U_b = e^{-i\pi f_x\cdot c}\cdot b\cdot \text{sinc}(\pi b f_x)$

Therefore $\quad U(x) \propto \mathcal{F}\{U(\xi)\}$

$$= a\ \text{sinc}(\pi a f_x)\cdot e^{+i\pi f_x\cdot c} + b\ \text{sinc}(\pi b f_x)\, e^{-i\pi f_x c}$$

$$I \propto |U(x)|^2$$

$$= \left(a\ \text{sinc}(\pi a f_x)\, e^{+i\pi f_x c} + b\ \text{sinc}(\pi b f_x)\, e^{-i\pi f_x c}\right)$$

$$\times \left(a\ \text{sinc}(\pi a f_x)\, e^{+i\pi f_x c} + b\ \text{sinc}(\pi b f_x)\, e^{-i\pi f_x c}\right)$$

$$= [a\ \text{sinc}(\pi a f_x)]^2 + [b\ \text{sinc}(\pi b f_x)]^2 + ab\ \text{sinc}(\pi a f_x)\ \text{sinc}(\pi b f_x) \times \left(e^{-j2\pi f_x c} + e^{+j2\pi f_x c}\right)$$

$$= a^2\ \text{sinc}^2(\pi a f_x) + b^2\ \text{sinc}^2(\pi b f_x) + 2ab\ \text{sinc}(\pi a f_x)\ \text{sinc}(\pi b f_x)\cdot \cos(2\pi f_x c)$$

$$f_x = \frac{x}{\lambda z} = \frac{1}{\lambda}\sin\theta$$

(i) $a = b$

$$\Rightarrow \quad I \propto 2a^2 \operatorname{sinc}^2(\pi a x/\lambda z)\left[1 + \cos\left(2\pi \cdot \frac{x}{\lambda z} c\right)\right]$$

$$= 4a^2 \operatorname{sinc}^2(\pi a x/\lambda z) \cdot \cos^2(\pi c x/\lambda z)$$

$\frac{x}{z} = \tan\theta \sim \sin\theta$
$\sim \theta$

$$\approx 4a^2 \operatorname{sinc}^2\left(\frac{\pi a}{\lambda} \sin\theta\right) \cdot \cos^2\left(\frac{\pi c}{\lambda} \sin\theta\right)$$

[redacted]

(ii) $a = 0$

$$I \propto b^2 \operatorname{sinc}^2(\pi b f_x) = b^2 \operatorname{sinc}^2(\pi b x/\lambda z) \approx b^2 \operatorname{sinc}^2\left(\frac{\pi b}{\lambda} \sin\theta\right)$$

This is the diffraction pattern for a single slit.

4)

A. The angular resolving power of a telescope is given by $\theta = \frac{1.22\lambda}{D}$.

For $\lambda = 5790 \text{Å}$ we have

$$D = \frac{1.22 \times 5790 \times 10^{-8}}{1 \times 10^{-6}} = 70.6 \text{ cm}$$

As this is larger than that for $\lambda = 5770 \text{Å}$, the diameter needed to separate the two stars should be at least 70.6 cm.

B. The chromatic resolving power of a grating is given by $\frac{\bar{\lambda}}{\Delta\lambda} = mN$,

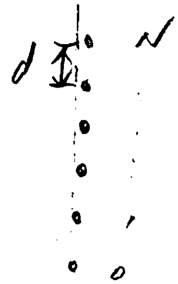
where N is the total number of rulings on the grating, $\bar{\lambda}$ the mean wavelength, and m the order of diffraction. One usually uses orders 1 to 3.

For $m = 1$, $N = \frac{\bar{\lambda}}{\Delta\lambda} = \frac{\frac{5770+5790}{2}}{5790-5770} = 289.$

For $m = 3$, $N = 96$.

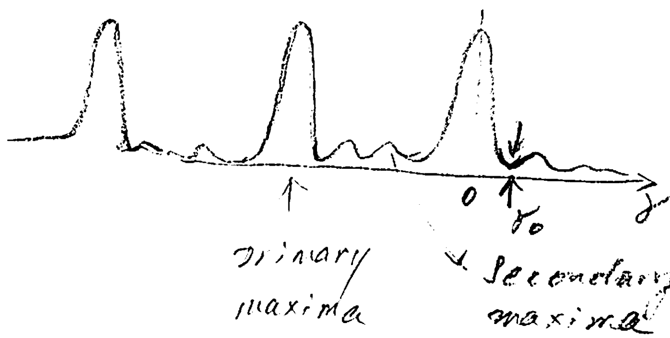
Hence a diffraction grating of 289 rulings is needed to separate the wavelengths present.

5) ideal grating



$$I \propto I_0 \left(\frac{\sin N\gamma}{N \sin \gamma} \right)^2 \qquad \gamma = \frac{1}{2} k d \sin\theta$$

$$I = 0 \text{ when } N\gamma = m\pi$$



$$N\gamma_0 = \pi$$

$$\rightarrow \quad N \times \frac{1}{2} \times \frac{2\pi}{\lambda} \times d \cdot \sin\theta = \pi$$

$$\rightarrow \quad \frac{N \cdot d \cdot \sin\theta_0}{\lambda} = 1$$

$$\sin\theta_0 = \frac{\lambda}{N \cdot d}$$

small angle approximation

$$\sin\theta_0 \simeq \theta_0 = \frac{\lambda}{Nd}$$

$$\text{Angular dimension} \approx 2\theta_0 = \frac{2\lambda}{Nd}$$

$$N = \frac{\ell}{d} = \frac{12 \times 10^3 \, \mu m}{1.5 \, \mu m} = 8000$$

① $$2\theta_0 = \frac{2 \times 0.530 \, \mu m}{8000 \times 1.5 \, \mu m} \approx 8.8 \times 10^{-5} \text{ (rad)}$$

$$\approx 5 \times 10^{-3} \text{ degree}$$

② Resolving power $\frac{\lambda}{\Delta\lambda} = N = 8000$ for the 1-st order diffraction