Scalar Diffraction Theory and Basic Fourier Optics

[Pedrotti Ch. 11, Ch. 12 and Ch. 21]

Scalar Electromagnetic theory:

$$\boxed{u(P,t) = Re[U(P)e^{-j\omega t}]} \qquad \text{monochromatic wave}$$

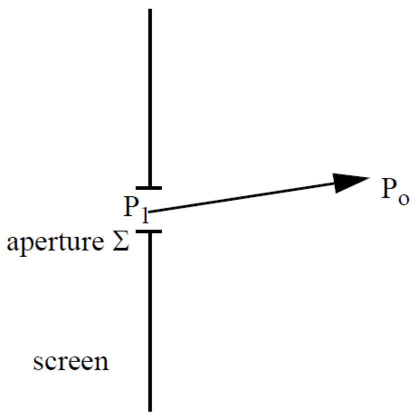
P : position $\quad t$: time $\quad \omega = 2\pi\nu$: optical frequency

u(P, t) represents the *E* or *H* field strength for a particular transverse polarization component

U(P) : represents the complex field amplitude

$U(P) = u(P)e^{-j\phi(P)}$ $\qquad u(P)$: real

Diffraction:



Approximations:

1. We impose the boundary condition on U, that $U = 0$ on the screen.
2. The field in the aperture Σ is not affected by the presence of the screen.

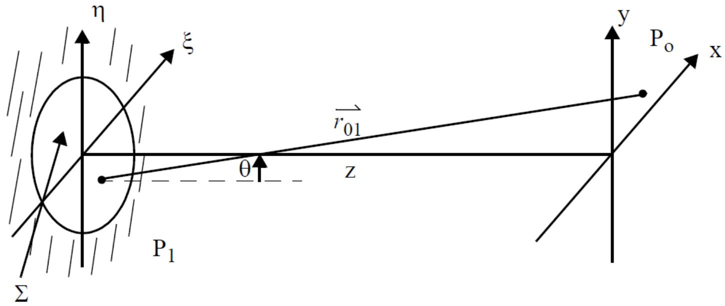
$$U(P_o) = \frac{1}{j\lambda}\iint_{\Sigma} U(P_1)\underbrace{\frac{\exp(jkr_{01})}{r_{01}}}_{\text{expanding spherical}}ds$$

$[r_{01} \gg \lambda]$

This equation expresses the Huygens-Fresnel principle: The observed field is expressed as a superposition of point sources in the aperture, with a weighting factor $\boxed{\frac{U(P_1)}{j\lambda}}$.

Fresnel approximation

Huygens-Fresnel integral in rectangular coordinates:



$$\boxed{r_{01} = [z^2 + (x-\xi)^2 + (y-\eta)^2]^{1/2}}$$

The Fresnel approximation involves setting: $r_{01} \cong z$ in the denominator, and

$$r_{01} \cong z\left[1 + \frac{1}{2}\frac{(x-\xi)^2}{z} + \frac{1}{2}\frac{(y-\eta)^2}{z}\right] \quad \text{in exponent}$$

This is equivalent to the paraxial approximation in ray optics.

$$U(x, y) = \frac{\exp(jkz)}{j\lambda z}\int\int_{-\infty}^{\infty} d\xi d\eta U(\xi, \eta)\exp\left\{\frac{jk}{2z}[(x-\xi)^2 + (y-\eta)^2]\right\} \qquad \text{(A)}$$

Let's examine the validity of the Fresnel approximation in the Fresnel integral. The next higher order term in exponent must be small compared to 1. So the valid range of the Fresnel approximation is:

$$z^3 \gg \frac{\pi}{4\lambda}[(x-\xi)^2 + (y-\eta)^2]^2_{max}$$

For field sizes of 1 cm, $\lambda = 0.5\mu m$, we find $z \gg 25$ cm.

Actually we should look at the effect on the total integral. Upon closer analysis, it is found that the Fresnel approximation holds for a much closer z. This is referred to as the "near-field region".

Farther out in z, we can approximate the quadratic phase as flat

$$z \gg \frac{k(\xi^2 + \eta^2)_{max}}{2}$$

This region is referred to as the "far-field" or Fraunhofer region.

$$U(x, y) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2 + y^2)}}{j\lambda z} \underbrace{\int\int d\xi d\eta U(\xi, \eta)\exp\left[-j\frac{2\pi}{\lambda z}(x\xi + y\eta)\right]}_{\mathcal{F}\{U(\xi, \eta)\}\big|_{f_x = \frac{x}{\lambda z}, f_y = \frac{y}{\lambda z}}}$$

Now this is exactly the Fourier transform of the aperture distribution with

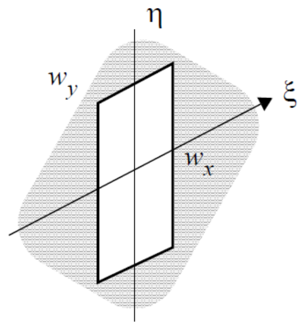
$$\boxed{f_x = \frac{x}{\lambda z}} \qquad \boxed{f_y = \frac{y}{\lambda z}}$$

The Fraunhofer region is farther out. For the field size of 1 cm, and $\lambda = 0.5\mu m$, we find the valid range of $z \gg 150$ meters!

Again, examining the full integral, Fraunhofer is actually accurate and usable to much closer distances.

Examples

A rectangular aperture, illuminated by a normally incident plane wave:



$$t_A = rect\left(\frac{\xi}{2w_x}\right)rect\left(\frac{\eta}{2w_y}\right)$$

With plane wave illumination, we have: $U(\xi, \eta) = t_A(\xi, \eta)$

$$\therefore \qquad U(x, y, z) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z} \mathcal{F}[U]\Bigg|_{\substack{f_x = \frac{x}{\lambda z} \\ f_y = \frac{y}{\lambda z}}}$$

$$= \frac{e^{jk\left[z + \frac{x^2+y^2}{z}\right]}}{j\lambda z} A \mathrm{sinc}\left(\frac{2w_x x}{\lambda z}\right) \mathrm{sinc}\left(\frac{2w_y y}{\lambda z}\right)$$

$$A \equiv 4w_x w_y$$

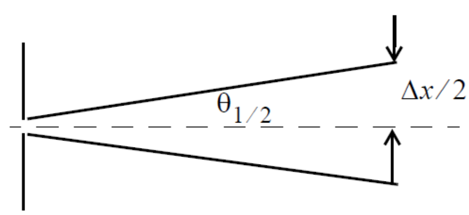
Recall $\boxed{\mathrm{sinc}(x) \equiv \frac{\sin \pi x}{\pi x}}$. The observable is intensity $I = |U|^2$.

$$I = \frac{A^2}{\lambda^2 z^2} \mathrm{sinc}^2\left(\frac{2w_x x}{\lambda z}\right) \mathrm{sinc}^2\left(\frac{2w_y y}{\lambda z}\right)$$

The width of the central lobe of the diffraction pattern is

$$\boxed{\Delta x = \frac{\lambda z}{w_x}}$$

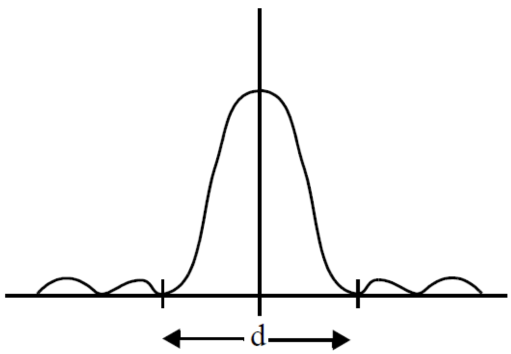
The diffraction half angle $\theta_{1/2} \cong \frac{\Delta x}{2} = \frac{\lambda}{2w_x}$

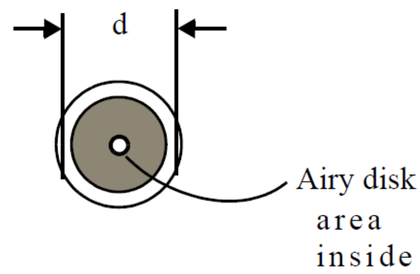


For a circular aperture with radius w : $t_A = circ\left(\frac{q}{w}\right)$ $\quad q^2 \equiv \xi^2 + \eta^2$ radial coordinates

In circular coordinates, we use the Fourier - Bessel transform: $\mathcal{B}\{U(q)\}$ gives immediately:

$$I(r) = \left(\frac{A}{\lambda z}\right)^2 \left[2\frac{J_1(kwr/z)}{kw(r/z)}\right]^2 \quad \text{"Airy pattern"}$$





$$\boxed{d = 1.22\frac{\lambda z}{w}} \quad \text{diameter of Airy disk}$$

An alternative approach:

To calculate the diffraction pattern of a circular aperture, we can choose y as the variable of integration. If R (**w** in the above figure) is the radius of the aperture, then the element of area is taken to be a strip of width dy and length $2\sqrt{R^2 - y^2}$.

The amplitude distribution of the diffraction pattern is then given by

$$U = Ce^{ikr_0} \int_{-R}^{R} e^{iky\sin(\theta)} 2\sqrt{R^2 - y^2}\, dy .$$

We introduce the quantities u and ρ defined by $u = y/R$ and $\rho = kR\sin(\theta)$. The integral then becomes

$$\int_{-1}^{+1} e^{i\rho u}\sqrt{1-u^2}\, du .$$

This is a standard integral. Its value is $\pi J_1(\rho)/\rho$ where J_1 is the Bessel function of the first kind, order one. The ratio $J_1(\rho)/\rho \to \frac{1}{2}$ as $\rho \to 0$. The irradiance/intensity distribution is therefore given by

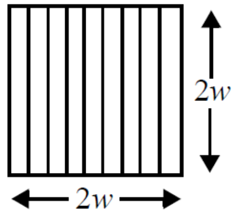
$$I = |U|^2 = I_0 \left[\frac{2J_1(\rho)}{\rho}\right]^2 .$$

The diffraction pattern is circularly symmetric and consists of a bright central disk surrounded by concentric circular bands of rapidly diminishing intensity. The bright central area is know as the Airy disk. It extends to the first dark ring whose size is given by the first zero of the Bessel function, namely, $\rho = 3.832$. The angular radius of the first dark ring is thus given by

$$\sin\theta = \frac{3.832}{kR} = \frac{1.22\lambda}{D} \approx \theta$$

which is valid for small values of θ (in radians). Here D=2R is the diameter of the aperture.

Diffraction grating (transmission)

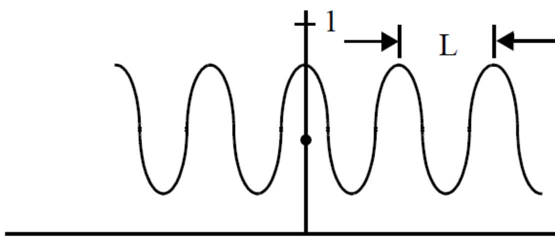


sinusoidal amplitude

$$t_A = \left[\frac{1}{2} + \frac{m}{2}\cos(2\pi f_0 \xi)\right]\mathrm{rect}\left(\frac{\xi}{2w}\right)\mathrm{rect}\left(\frac{\eta}{2w}\right)$$

m: peak to peak amplitude change $0 \le m \le 1$

f_0: grating spatial frequency

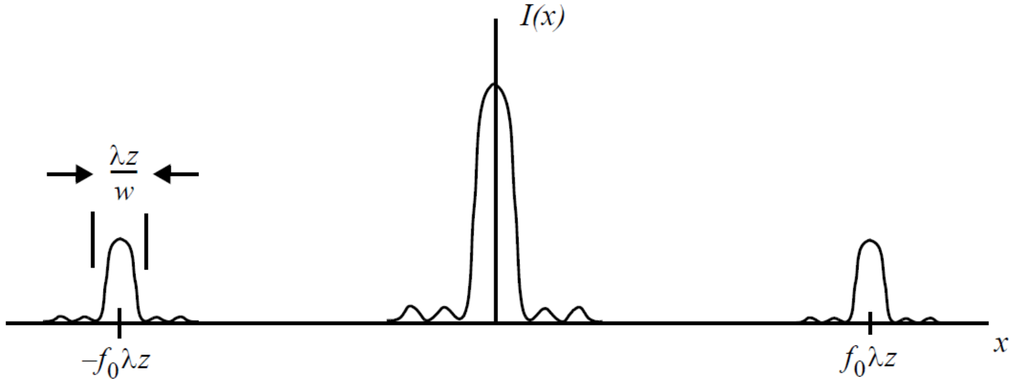


grating period

$$L = \frac{1}{f_0}$$

By convolution, the diffracted amplitude is

$$F\left[\frac{1}{2} + \frac{m}{2}\cos(2\pi f_0 \xi)\right] \otimes F\left[rect\frac{\xi}{2w} rect\frac{\eta}{2w}\right]$$



$$I(x,y) \cong \left(\frac{A}{2\lambda z}\right)^2 \mathrm{sinc}^2\frac{2wy}{\lambda z}\left\{\mathrm{sinc}^2\left(\frac{2wx}{\lambda z}\right) + \frac{m^2}{4}\mathrm{sinc}^2\left[\frac{2w}{\lambda z}(x + f_0\lambda z)\right] + \frac{m^2}{4}\mathrm{sinc}^2\left[\frac{2w}{\lambda z}(x - f_0\lambda z)\right]\right\}$$

We have neglected interference terms between orders.

Compared to the square aperture, which has the central peak with intensity I_o, we now have:

$$\frac{1}{4}I_0 \;:\; \text{zero order}$$

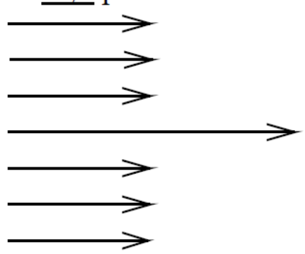
$$\frac{m^2}{16}I_0 \;:\; \pm 1 \text{ order}$$

The "resolving power" of the grating $R \equiv \dfrac{\text{peak separation}}{\text{peak width}}$

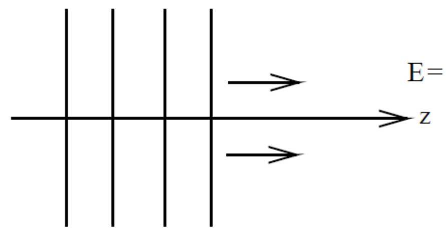
$$R = \frac{f_0 \lambda z}{\lambda z / w} = f_0 w = \frac{w}{L} = [\text{\# grating periods}]$$

Wave Optics of Lenses

Set of rays parallel to axis

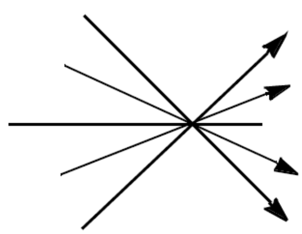


Plane Wave

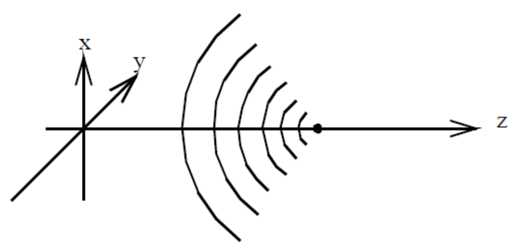


$$E = E_\circ \cos(kz - \omega t)$$
$$k = \frac{2\pi}{\lambda}$$
$$\omega = 2\pi f$$

Rays converging to a focus



converging spherical wave



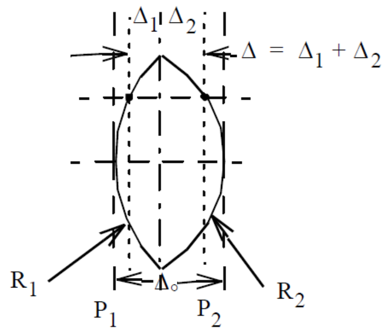
At a given z-plane, the spherical wave has constant phase around circles. The form of the spherical wave is $\cos\left[-\dfrac{k(x^2+y^2)}{2z_\circ}\right]$ for a spherical wave converting to the point $z_\circ$ on the axis. A lens modifies the wave front, for example from planar to spherical.

How does this happen?

Optical Path Difference

Optical waves travel more slowly in the glass since $n > 1$. In glass, the wave is delayed by an amount as if it travelled a distance nl in free space. If $l = l(x,y)$ [or $n = n(x,y)$] then the delay varies with (x,y) so the wavefront gets distorted.

We can analyze the lens in terms of its phase-delay. The light propagates in the glass as $\cos(knz) = \cos\phi$, where $\phi = knz$ is the phase delay.



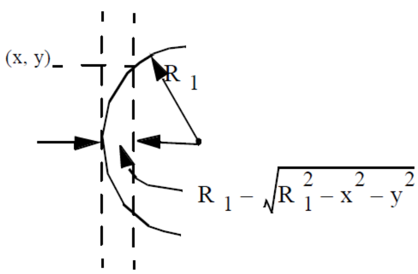
In propagating from plane P_1 to P_2, the light travels a distance $\Delta = \Delta_1 + \Delta_2$ in the glass and a distance $\Delta_\circ - \Delta$ in air, where $\Delta_\circ$ is the thickness at the thickest part of the lens. The phase delay depends on (x, y):

$$\phi(x, y) = kn\Delta(x,y) + k[\Delta_\circ - \Delta(x, y)]$$

$$= k\Delta_\circ + k(n-1)\Delta(x,y)$$

We can calculate Δ, assuming spherical surfaces. Recall the sign convention for the surface radii:

positive radius negative radius



From this diagram, we can readily obtain

$$\Delta(x, y) = \Delta_\circ - \left[R_1 - \sqrt{R_1^2 - x^2 - y^2}\right] + \left[R_2 - \sqrt{R_2^2 - x^2 - y^2}\right]$$

$$= \Delta_\circ - R_1\left[1 - \sqrt{1 - \left(\frac{x^2 + y^2}{R_1^2}\right)}\right] + R_2\left[1 - \sqrt{1 - \left(\frac{x^2 + y^2}{R_2^2}\right)}\right]$$

In the paraxial approximation $(x^2+y^2) \ll R_{1,2}^2$, so

$$\sqrt{1-\left(\frac{x^2+y^2}{R_{1,2}^2}\right)} \cong 1-\left(\frac{x^2+y^2}{2R_{1,2}^2}\right)$$

, thus

$$\Delta \cong \Delta_\circ - \left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

This gives a phase delay:

$$\phi(x, y) = k\Delta_\circ + k(n-1)\left[\Delta_\circ - \left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]$$

Apart from the constant delay $kn\Delta_\circ$, the phase delay is:

$$\boxed{\phi(x, y) = -k(n-1)\left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)}$$

A plane wave incident on the lens has a constant phase. After passing through the lens, the phase is given above. This has the form of a spherical wave, converging to a point at a distance f, where

$$\boxed{\frac{1}{f} = (n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)},$$

f is the focal length of the lens. This expression is identical to what we found from the ray optics analysis.

Diffraction Theory of a Lens

We have previously seen that light passing through a lens experiences a phase delay given by:

$$\varphi(x,y) = \exp\left[-jk(n-1)\left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right] \quad \text{(neglecting the constant phase)}$$

The focal length, f is given by:

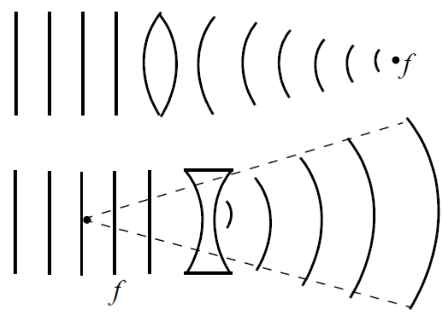
$$\boxed{\frac{1}{f} \equiv (n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} \quad \text{The "lens makers formula"}$$

The transmission function is now:

$$\boxed{\varphi(x,y) = \exp\left[-j\frac{k}{2f}(x^2+y^2)\right]}$$

This is the paraxial approximation to the spherical phase

Note: the incident plane-wave is converted to a spherical wave converging to a point at f behind the lens (f positive) or diverging from the point at f in front of lens (f negative).



Diffraction from the lens pupil

Suppose the lens is illuminated by a plane wave, amplitude A. The lens "pupil function" is $P(x,y)$.

The full effect of the lens is $U_l'(x,y) = \varphi(x,y)P(x,y)$

$$U_l'(x,y) = P(x,y)\exp\left[-j\frac{k}{2f}(x^2+y^2)\right]$$

We now use the Fresnel formula to find the amplitude at the "back focal plane" $z = f$

$$U_f(u,v) = \frac{\exp\left[j\frac{k}{2f}(u^2+v^2)\right]}{j\lambda f} \times e^{jkf}\int\int_{-\infty}^{\infty} dxdy\, U_l'(x,y)\exp\left[j\frac{k}{2f}(x^2+y^2)\right]\exp\left[-j\frac{2\pi}{\lambda f}(xu+yv)\right]$$

The phase terms that are quadratic in x^2+y^2 cancel each other.

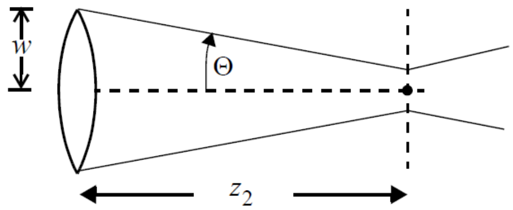
$$U_f(u,v) = \frac{\exp\left[j\frac{k}{2f}(u^2+v^2)\right]}{j\lambda f}e^{jkf}\int\int_{-\infty}^{\infty} dxdy\, P(x,y)\exp\left[-j\frac{2\pi}{\lambda f}(xu+yv)\right] \quad \text{(B)}$$

This is precisely the Fraunhofer diffraction pattern of P ! Note that a large z criterion *does not* apply here.

The focal plane amplitude distribution is a Fourier transform of the lens pupil function *P(x,y)*, multiplied by a quadratic phase term. However, the intensity distribution is exactly

$$I_f(u, v) = \frac{A^2}{\lambda^2 f^2} |\mathcal{F}[P(x, y)]|^2 \qquad f_x = \frac{u}{\lambda f} \qquad f_y = \frac{v}{\lambda f}$$

Example: a circular lens, with radius *w*



$$P = \mathrm{circ}\left(\frac{q}{w}\right) \qquad (q^2 = x^2 + y^2)$$

$$\text{let} \quad h(r) = \mathcal{F}[P(\lambda z_2 q)] = \mathcal{F}\left[\mathrm{circ}\left(\frac{\lambda z_2 q}{w}\right)\right] \qquad (r^2 = u^2 + v^2)$$

$$= \frac{A}{\lambda z_2}\left[2\frac{J_1(2\pi wr/\lambda z_2)}{2\pi wr/\lambda z_2}\right]$$

$$|h(r)|^2 = \frac{A^2}{\lambda^2 z_2^2}\left[2\frac{J_1(2\pi wr/\lambda z_2)}{2\pi wr/\lambda z_2}\right]^2$$

The spot diameter is $d = 1.22\frac{\lambda f}{w} = 1.22\frac{\lambda}{\theta}$

The resolution of the lens as defined by the "Rayleigh" criterion is $d/2 = 0.61\lambda/\theta$**. For a small angle θ,** $d/2 = 0.61\lambda/\sin\theta = 0.61\frac{\lambda}{NA}$**.**