Prisms [Reading Pedrotti^3 Ch. 3-3]

Dispersion
This variation of the refractive index with the wavelength or frequency of the light is called dispersion. Dispersion is a property of all transparent materials.

White light passing through a prism is broken down into its constituent colors.

<table>
<thead>
<tr>
<th>Color</th>
<th>Frequency</th>
<th>Wavelength</th>
</tr>
</thead>
<tbody>
<tr>
<td>violet</td>
<td>668–789 THz</td>
<td>380–450 nm</td>
</tr>
<tr>
<td>blue</td>
<td>631–668 THz</td>
<td>450–475 nm</td>
</tr>
<tr>
<td>cyan</td>
<td>606–630 THz</td>
<td>476–495 nm</td>
</tr>
<tr>
<td>green</td>
<td>526–606 THz</td>
<td>495–570 nm</td>
</tr>
<tr>
<td>yellow</td>
<td>508–526 THz</td>
<td>570–590 nm</td>
</tr>
<tr>
<td>orange</td>
<td>484–508 THz</td>
<td>590–620 nm</td>
</tr>
<tr>
<td>red</td>
<td>400–484 THz</td>
<td>620–750 nm</td>
</tr>
</tbody>
</table>

The above visible spectrum does not show all the colors seen in nature. Many of the colors we see are a mixture of wavelengths. For practical purposes, most natural colors can be reproduced using three primary colors. They are red, green, and blue (R-G-B) for direct source viewing such as TV and computer monitors. For inks used in printing, the primary colors are cyan, yellow, and magenta (C-Y-M) (& black CYMK).

Index of refraction as a function of wavelength for various transparent solids.

(image source: Wikipedia.org)
Rainbows
Rainbows are a spectacular example of dispersion – by drops of water. You can see rainbows when you look at falling water droplets with the Sun behind you. Red and violet rays are bent by spherical water droplets and are reflected off the back surfaces (total internal reflection). Red is bent the least and so reaches the observer’s eyes from droplets higher in the sky, as shown in the diagram. Thus the top of the (primary) rainbow is red.

(image source: http://rebeccapaton.net/rainbows/formatn.htm & wikipedia.org)

Diamond
Diamonds achieve their brilliance from a combination of dispersion and total internal reflection. Because diamonds have a very high index of refraction of about 2.4, the critical angle for total internal reflection is only 25°. The light dispersed into a spectrum inside the diamond therefore strikes many of the internal surfaces before it strikes one at less than 25° and emerges. After many such reflections, the light has traveled far enough that the colors have become sufficiently separated to be seen individually and brilliantly by the eye after leaving the crystal.
Angular deviation of a prism [Pedrotti^3 Ch. 3-3]

Figure 5.56 Geometry of a dispersing prism.

The total deviation:

\[ \delta = (\theta_2 - \theta_1) + (\theta_{2a} - \theta_{1a}) \]

Since \( \alpha = \theta_{2a} + \theta_{1a} \),

\[ \delta = \theta_{2a} + \theta_{1a} - \alpha \]

From Snell's law

\[ \theta_{2a} = \sin^{-1}(n \sin \theta_{2a}) = \sin^{-1}\left[\left(\sin \alpha\right)(n^2 - \sin^2 \theta_{1a})^{1/2} - \sin \theta_{1a} \cos \alpha\right] \]

The deviation is then

\[ \delta = \theta_{2a} + \sin^{-1}\left[\left(\sin \alpha\right)(n^2 - \sin^2 \theta_{1a})^{1/2} - \sin \theta_{1a} \cos \alpha\right] - \alpha \]

\( \delta \) Increases with \( n \). For visible light, \( n \) increases as frequency increases. Therefore, blue light deviates more than red light.

At minimum deviation angle

\[ \frac{d\delta}{d\theta_{2a}} = 1 + \frac{d\theta_{2a}}{d\theta_{1a}} = 0 \Rightarrow \frac{d\theta_{2a}}{d\theta_{1a}} = -1 \]

Also

\[ \alpha = \theta_{2a} + \theta_{1a} \Rightarrow d\alpha = -d\theta_{2a} \]

Taking derivatives of Snell's law at the two interface, we get,

\[ \cos \theta_{1a} d\theta_{1a} = n \cos \theta_{1a} d\theta_{1a} \]
\[ \cos \theta_{2a} d\theta_{2a} = n \cos \theta_{2a} d\theta_{2a} \]

therefore

\[ \frac{\cos \theta_{1a}}{\cos \theta_{2a}} = \frac{\cos \theta_{1a}}{\cos \theta_{2a}} \text{ or } \frac{1 - \sin^2 \theta_{2a}}{1 - \sin^2 \theta_{1a}} = \frac{n^2 - \sin^2 \theta_{1a}}{n^2 - \sin^2 \theta_{2a}} \]

Since \( n \neq 1 \), we have \( \theta_{1a} = \theta_{2a} \), and \( \theta_{2a} = \theta_{1a} \)

3
At the minimum deviation angle $\delta_m$,
\[
\theta_d = \frac{(\delta_m + \alpha)}{2},
\]
\[
\theta_l = \frac{\alpha}{2},
\]
and
\[
\sin \left[ \frac{(\delta_m + \alpha)}{2} \right] = \frac{n \sin \alpha}{2}
\]
One of the most accurate technique for determining the refractive index.

The deviation increases with decreasing index $n$. For most materials $n$ increases with decreasing $\lambda$. This is the basis for the splitting of white light into colors by the prism.
Selected Applications of Prisms

Pedrotti\textsuperscript{3} Figure 3-18. Image manipulation by reflecting prisms. (a) Right-angle prism. (b) Dove prism. (d) Penta prism. (d) Porro prism.

Right Angle Prism

Porro prism

Erecting Prisms
Most telescopes produce an inverted image (both U-D, L-R) to the eye. Erecting prisms re-invert the image to the proper orientation.

2 porro prisms used together. Generally contacted.
Lateral displacement through a plane parallel glass plate

\[
d = \frac{t \sin(\theta - \theta')}{\cos(\theta')}
\]

Using the trigonometric identity \( \sin(\theta - \theta') = \sin \theta \cos \theta' - \cos \theta \sin \theta' \); we get:

\[
d/t = \sin \theta [1 - \cos \theta'/(n \cos \theta')]
\]

where:

\[
\cos \theta' = \sqrt{1 - (\sin^2 \theta)/n^2}
\]

This can be used to laterally displace an image. One application of this very simple device is in a specialized high speed camera. The film has to move so fast that it is driven continuously (rather than actually stopping briefly for each frame as in a conventional camera). A rotating plate is used to make the image track the moving film during exposure of a given frame to prevent blur.

But the plate introduces aberrations.

- Chromatic effect: longitudinal and lateral displacements depend on \( n \) which is \( \lambda \) dependent.
- For a plate used in convergent or divergent light, the amount of displacement is greater for larger angles which gives spherical aberration.

Plane parallel plate placed in between a lens and its focus:

A simple calculation based on the paraxial approximation shows that the focus is displaced by amount \( \frac{n-1}{n} t \). However, at steeper incidence angles, the focal shift becomes a function of the incidence angle, which leads to spherical aberration.
Thick Lens [Optional, won't appear in exams]

**Principle plane:** the plane on which the extension lines of the ray incident from the first focus and the ray emerged from the lens intercept.

**Secondary Plane:** the same as the principle plane except that the ray is from the second focus.

**First principal point** \( H_1 \): the intersection of the Principle plane and the optical axis.

**Second principal point** \( H_2 \): the intersection of the secondary plane and the optical axis.

**Nodal points** \( N_1 \) and \( N_2 \): the interception of the incident and emerged rays which pass the optical center with optical axis.

**Cardinal Points:** the two focal, two principal and two nodal points.
Thick Lens Formula

Single Lens

If consider the thick lens as the combination of two spherical refracting surface separated by a distance $d_i$, the result is

\[
\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}
\]

\[
\frac{1}{f} = (n_i-1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n_i-1)d_i}{n_iR_1R_2}\right]
\]

Note that $s_o$, $s_i$ and $f$ are measured from the first and second principal planes. Also the distance of the principal points and the vertices $\overline{V_1H_1} = h_1$ and $\overline{V_2H_2} = h_2$ are

\[
h_1 = -\frac{f(n_i-1)d_i}{R_2n_i}
\]

\[
h_2 = -\frac{f(n_i-1)d_i}{R_1n_i}
\]

which are positive when the principal points lie to the right of their respective vertices.
Chromatic Aberration [Reading Pedrotti^3 Ch. 3-5 & Ch. 20-7]

1. Due to the dependency of the refraction index on frequency. Lights with different wavelengths have different focuses.
2. A·CA: axial chromatic aberration.
3. L·CA: lateral chromatic aberration.

Figure 6.36 Axial chromatic aberration.

Figure 6.37 Lateral chromatic aberration.

Thin Achromatic Doublets

Purpose: to bring the focus of the red and blue lights together by a combination of two thin lens separated by a distance \( d \).

\[
\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}
\]

Let

\[
\frac{1}{f_1} = (n_1-1)(\frac{1}{R_{11}} - \frac{1}{R_{12}}) = (n_1-1)\rho_1, \quad \frac{1}{f_2} = (n_2-1)(\frac{1}{R_{21}} - \frac{1}{R_{22}}) = (n_2-1)\rho_2
\]

then

\[
\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} = (n_1-1)\rho_1 + (n_2-1)\rho_2 - d(n_1-1)\rho_1 (n_2-1)\rho_2
\]

Let the focus of red light be \( f_R \) and blue light \( f_B \). What we want is \( \frac{1}{f_R} - \frac{1}{f_B} \).

This leads to

\[
(n_{1R}-1)\rho_1 + (n_{2R}-1)\rho_2 - d(n_{1R}-1)\rho_1 (n_{2R}-1)\rho_2 = (n_{1B}-1)\rho_1 + (n_{2B}-1)\rho_2 - d(n_{1B}-1)\rho_1 (n_{2B}-1)\rho_2
\]

Case 1: Select \( d=0 \), we have
Consider yellow as the center of the spectrum from blue to red.

Let the focus of yellow light be \( f'_{Y} \), then

\[
\frac{1}{f'_{1Y}} = (n_{1Y} - 1)\rho_{1}, \quad \frac{1}{f'_{2Y}} = (n_{2Y} - 1)\rho_{2} \quad \Rightarrow \quad \frac{\rho_{1}}{\rho_{2}} = \frac{(n_{2Y} - 1)f'_{2Y}}{(n_{1Y} - 1)f'_{1Y}}.
\]

Therefore,

\[
f'_{2Y} = \frac{(n_{2B} - n_{2R})(n_{2Y} - 1)}{(n_{1B} - n_{1R})(n_{1Y} - 1)}.\]

Definition:

1. Dispersive power: \( \frac{n_{B} - n_{R}}{n_{Y} - 1} \)

2. Dispersive index, or V-number, or Abbe number: \( V = \frac{n_{Y} - 1}{n_{B} - n_{R}} \)

Thus,

\[
f'_{2Y} = -\frac{V_{1}}{V_{2}} \Rightarrow f'_{1Y}V_{1} + f'_{2Y}V_{2} = 0
\]

Case 2: Select \( n_{1} = n_{2} \), then

\[
d = \frac{1}{n_{B} + n_{R} - 2} \left( \frac{1}{\rho_{1}} + \frac{1}{\rho_{2}} \right) = \frac{(f'_{1Y} + f'_{2Y})(n_{Y} - 1)}{n_{B} + n_{R} - 2}.
\]

If \( n_{Y} = \frac{n_{B} + n_{R}}{2} \), we have

\[
d = \frac{f'_{1Y} + f'_{2Y}}{2}
\]
**Aberrations (Additional online video resources)**


Chromatic Aberration: Youtube Clip [http://www.youtube.com/watch?v=yOR4WHgRfvI&NR=1](http://www.youtube.com/watch?v=yOR4WHgRfvI&NR=1)

Coma Aberration: Youtube Clip [http://www.youtube.com/watch?v=EXmaY2txEBo&NR=1](http://www.youtube.com/watch?v=EXmaY2txEBo&NR=1)

MiniTutorial of Geometrical Aberrations (Seidel Aberrations):

Youtube clips 1. [http://www.youtube.com/watch?v=wzEQX1tMLdY](http://www.youtube.com/watch?v=wzEQX1tMLdY)  
2. [http://www.youtube.com/watch?v=TWGXKj_RIs0](http://www.youtube.com/watch?v=TWGXKj_RIs0)