**Exchange Enhancement of Pauli susceptibility**

**These are the problems 5 and 6 of Chapter 11 of the 8th edition or Chapter 14 of the 7th edition (Diamagnetism and Paramagnetism). Those of who can take a look at the derivation-it is quite simple. For the exam please review Eq. 1, Eq. 12, and the discussions (numerics) following Eq. 12.**

For noninteracting itinerant electrons the Pauli paramagnetic susceptibility was derived earlier by calculating the number of up and down (moment parallel and antiparallel to the external field Ba). It is given by

$χ\_{Pauli}=χ\_{P}^{0}\_{ }^{ }=μ\_{B}^{2}D\left(ε\_{F}\right)$ (1)

where *µB*and *D*(*εF*)=3*N*/2 *εF* are respectively the Bohr magneton and electron energy density of states at the Fermi energy for *N* electrons. We have a superscript *0* because this is for noninteracting electrons.

For interacting electrons we want to show that

$χ\_{P}=\frac{χ\_{P}^{0}}{1-α}$ (2)

where the quantity *α* = *VD*(*εF*)/2 is called the exchange enhancement parameter which depends on the strength of electron-electron interaction (*V*) and *D*(*εF*).

To derive this we will use the energy variation principle instead of calculating directly the number of up and down electron in the presence of external field.

ENERGY APPROACH.

First we will work with non-interacting electrons and derive Eq.(1). We will use the fact that for free electrons in 3 dimension, *N*~ *εF* 3/2 and the total kinetic energy *Ekin*~ *εF* 5/2. This gives Ekin~*N* 5/3.

Now let us assume that in the presence of an external field we have *N*(1+*ζ*)/2 and *N*(1-*ζ*)/2 electrons with moment parallel () and antiparallel () to the field respectively. Clearly the magnetic moment *M*= *NζµB*. *ζ*  is called the spin polarization parameter. We have to find *ζ* using minimum energy principle.

The kinetic energy of up electrons

$E\_{kin,\uparrow }=\frac{3}{10}Nε\_{F}\left(1+ζ\right)^{5/3} and E\_{kin,\downright }=\frac{3}{10}Nε\_{F}\left(1-ζ\right)^{5/3}$ (3)

You should check this. When

$N\_{\uparrow }=N\_{\downright } ; ζ=0 ; E\_{kin,\uparrow }=E\_{kin,\downright }=\frac{3}{10}Nε\_{F} ; E\_{kin,\uparrow }+E\_{kin,\downright }=\frac{3}{5}Nε\_{F}$ (4)

This is old well known result. Now add the kinetic energies and magnetic energies for up and down electrons to get the total energy as a function of the spin polarization parameter *ζ*.

$$E\_{tot}\left(ζ\right)= \frac{3}{10}Nε\_{F}\left(1+ζ\right)^{5/3}-μ\_{B}B\left(N/2\right)\left(1+ζ\right)+\frac{3}{10}Nε\_{F}\left(1-ζ\right)^{5/3}-μ\_{B}B\left(N/2\right)\left(1-ζ\right)$$

 (5)

For small value of B, the spin polarization parameter *ζ*  is small. We expand the kinetic energies as a power series in *ζ* . The odd terms cancel and the even terms remain. Keeping up to the quadratic term we find

$E\_{tot}\left(ζ\right)=\frac{3}{5}Nε\_{F}+\frac{1}{3}Nε\_{F}ζ^{2}-Nμ\_{B}Bζ$ (6)

Minimizing this with respect to *ζ* , we find

$ζ=\frac{3}{2ε\_{F}}μ\_{B}B ;M=Nζμ\_{B}=\frac{3N}{2ε\_{F}}μ\_{B}^{2}B=χ\_{P}^{0}B$ (7)

This gives the Pauli susceptibility for noninteracting electrons as obtained before.

$χ\_{P}^{0}=μ\_{B}^{2}D\left(ε\_{F}\right)$` (8)

PAULI SUSCEPTIBILITY FOR INTERACTING ELECTRONS (NEW RESULT)

We use a simple model. Each electron interacts with another electron of same spin with an attractive interaction (exchange effect due to Pauli principle) of strength *V*. We add to Eq. 7 two additional energy lowering terms coming from *V*. These are:

$-\frac{1}{2}V\left(\frac{N}{2}\left(1+ζ\right)\right)^{2}-\frac{1}{2}V\left(\frac{N}{2}\left(1-ζ\right)\right)^{2}$ (9)

The factor of ½ takes care of double counting, electron I interacts with electron j and vice versa. In the total interaction energy we should count this interaction only once. Note that the term linear in *ζ* cancels and only constant and quadratic ζ terms survive, modifying the kinetic energy term. The total energy including the exchange terms (up to quadratic terms in *ζ*) now becomes

$E\_{tot}\left(ζ\right)=\frac{3}{5}Nε\_{F}-\frac{1}{4}VN^{2}+\left[\frac{1}{3}Nε\_{F}-\frac{1}{4}VN^{2}\right]ζ^{2}-Nμ\_{B}Bζ$ (10)

Note that since total energy has to be extensive i.e. scale as *N*, the interaction parameter *V* in the present model should go as 1/*N*. We can define *V*=*v*/*N*. Minimizing the above total energy with respect to the parameter ζ and substituting this value in the expression for *M*, we get

$M=\frac{Nμ\_{B}^{2}}{\frac{2}{3}ε\_{F}\left(1-\frac{3}{2}\frac{1}{ε\_{F}}\frac{1}{2}NV\right)}B$ (11)

**Using the expression for *D*(*εF*)=3*N*/2 *εF* , we obtain**

$χ\_{P}=\frac{χ\_{P}^{0}}{1-α}$ **(12)**

**Where the exchange enhancement parameter *α* = *VD*(*εF*)/2, the desired result. Note that *D*(*εF*) scales as *N* and *V* sacles as 1/N. So the parameter is independent of the total number of electrons.**

**For alkali metals: *εF*~3 eV. *V*=*v*/*N* with *v*~1 eV. This gives**

$$α=\frac{3v}{4ε\_{F}}=0.25$$

**This gives**

$$\frac{χ\_{P}}{χ\_{P}^{0}}=\frac{1}{1-0.25}=1.33$$