Experiment 2

Random Error and Basic Statistics

Homework 2: turn in the second week of the experiment. **This is a difficult homework set, but a very important one. Allow adequate time to do a good job.** These are exercises on formulas you will use all term.

Read sections 3.1-3.10 of Taylor (you can skip 3.2). Read the handout on the important things in uncertainty calculations. Look carefully at the definition of independence in the handout.

Do the problems below. They and the analysis and discussion requested below will help prepare you for the uncertainty calculations needed for this lab, and indeed for uncertainty calculations all term. Using a spreadsheet is usually easier than a calculator. If you decide to do the additions in quadrature on a calculator, note that the conversion from rectangular to polar coordinates automatically calculates $\sqrt{(x^2 + y^2)}$ for given x and y.

1)   If x has been measured as 4.0 ± 0.1 cm, what should I report for x^2 and x^3? Give percent and absolute uncertainties, as determined by rule (3.10) for a power.

2) A student measures $a = 50 \pm 5$, $c = 60 \pm 2$, $e = 5.8 \pm .3$, all in cm, and calculates the sums a+c and a+e. Assuming the original errors were independent and random, find the uncertainties in her answers (using rule 3.16, "errors add in quadrature"). If she has reason to think the original errors are *not* independent, find the uncertainties in her answers (using rule 3.17, "errors add directly"). Summarize your calculations in a table. Useful headings might be Sum, Value, δq(independent), δq(not independent). Indicate with an asterisk those cases in which the second uncertainty (in c or e) can be entirely ignored, assuming the uncertainties are needed with only one significant figure. Comment on the comparative sizes of the uncertainties in each case.

3) A student makes the following measurements

$a = 5 \pm 1$ cm, $b = 18 \pm 2$ cm, $c = 12 \pm 1$ cm, $t = 3.0 \pm 0.5$ s, $m = 18 \pm 1$ gm

Calculate the fraction uncertainty of each quantity (q=a,b,c,t,m) and put this in a table with headings

        q   value   δq   δq /q (%).

Compute the quantities q=a+b+c and q=m b/ t , their uncertainties, and percentage uncertainties for the two cases of independent uncertainties, and *not* independent uncertainties. Show the formulas you will use, and arrange your results in a table with headings as below.

		Independent		not independent		
q	value	δq	$\delta q /q$ (%)	δq	$\delta q /q$ (%)	most important variable

For the "most important variable", indicate **which variable contributes most strongly** to the uncertainty for q. Why did the importance of δb change between the q=a+b+c and the q = m b/ t cases?

Explain whether the uncertainty of a+b-c will be the same, or different from, the uncertainty of a+b+c. From this answer, explain whether the fractional uncertainty of a+b+c will be larger or smaller than the fractional uncertainty of a+b-c, and why.

4) A student is studying the properties of a resistor. She measures the current flowing through the resistor and the voltage across it as

$$I = 2.10 \pm 0.02 \text{ amps} \quad \text{and} \quad V = 1.02 \pm 0.01 \text{ volts.}$$

a) What should be her calculated value for the power delivered to the resistor, P = IV, with its uncertainty? b) What for the resistance R = V/I (in ohms)? Assume the original uncertainties are independent. With I in amps and V in volts, the power P comes out in watts and the resistance in ohms. Start by evaluating the fractional uncertainty of I, and V in percent, then calculate $\delta P/P$ in percent following the example on p 62. Show algebra, not just numbers. Finally, derive δP from $\delta P/P$. For b), see if you can avoid repeating the entire calculation.

5) In an experiment on the conservation of angular momentum, a student needs to find the angular momentum L of a uniform disc of mass M and radius R as it rotates with angular velocity ω. She makes the following measurements:
$$M = 1.10 \pm .01 \text{ kg}, \quad R = .250 \pm .005 \text{ m}, \quad \omega = 21.5 \pm 0.4 \text{ rad/s}$$
And then calculates L as $L = \frac{1}{2} MR^2 \omega$. (The factor $\frac{1}{2} MR^2$ is just the moment of inertia of the uniform disc.) What is her answer for L with its uncertainty? (Consider the three original uncertainties independent and remember that the fractional uncertainty in R^2 is twice that in R.) For this and any calculations in the lab involving products proceed in the same way as you did for 4).

1. Introduction

Contrary to the naïve expectation, the experiments in physics typically involve not only the measurements of various quantitative parameters of nature. In almost all the situations the experimentalist has also to present an argument showing how confident she is about the numeric values obtained. Among other things, this confidence in the validity of the presented numeric data strongly depends on the accuracy of the measurement procedure. As a simple example, it is impractical to measure a mass of a feather using the scale from the truck weighing station, which is hardly sensitive to the weight less than a few pounds.

Another challenge has to be met when the scientist tries to compare the results of her experiment with the data from another experiments, or with the theoretical predictions. Since the conditions of the measurement almost always vary from an experiment to an experiment, and since they are also different from the idealized situation of the theoretical model, the compared values most likely will not match each other exactly. The task is then to figure out how important the factors creating this discrepancy are. If these factors are stable (**do not change from measurement to measurement**) and noticeable, they are called **systematic errors**. If, on the contrary, these factors are variable, they are called random, or statistical, errors. An important fact is that the uncertainty due to the random errors can be reduced by increasing the number of measurements to average over the random variations.

The main purpose of this experiment is to introduce you to methods of dealing with the uncertainties of the experiment. The basic procedures to correctly estimate the uncertainty in the knowledge of the measured value (*the error of the measurement*) include:

- correct treatment of the random errors and systematic errors of the experiment (*Taylor Chapters 4 and 5*);
- rounding off the insignificant digits in the directly measured and calculated quantities (Chapter 2 Taylor and the Reference Guide).

2. Goals

1. Understand basic statistical measures of uncertainty.
2. Learn when standard deviation vs. standard deviation of the mean is appropriate.
3. Distinguish between systematic and random errors.
4. Learn one method for estimating the random errors.
5. Learn how one way to estimate systematic errors.
6. Test for a statistically significant difference from an expected value.
7. To measure a time with accuracy better than half a percent.

3. Preliminary discussion (10-15 minutes).

Before the lab, you are asked to read and understand the theoretical material for this lab (Exp 2 and *Taylor*). Before the experiment starts, your group needs to decide which information will be relevant to your experiment. Discuss what you will do in the lab and what preliminary knowledge is required for successful completion of each step. Warning: this lab is a bit short, and the next is a bit long. **Don't leave early: but start Exp3.**

Think hard about organizing your work in an efficient way. What measurements will you need to make? Go through your lab manual with a highlighter, then make checklist of the needed measurements. What tables or spreadsheets will you need to make to organize the calculations data? How should you use *Kgraph* to expedite your calculations and unit conversions (when necessary)? What tables will you need to summarize your analysis

and conclusions from the data? This lab will have more explicit reminders about tables than future labs, but you should be thinking about this organization of data taking, data reduction, and summarization in every lab.

Questions for the preliminary discussion

You should write your own answer to each question in your lab book, but leave space to change it after discussion. If you do change your answer, say why.

3.1 Calculators and computers typically return results with as many digits as possible, including digits well beyond our measurement uncertainty. What procedure will you follow to systematically get rid of these insignificant digits?
3.2 What tables will you need to record your input data? What tables will you need to summarize your results? Scan the lab, write down your tables, then see Appendix.

Random Uncertainties

4.1 Introduction In this experiment we ask you to perform a simple repetitive measurement in order to investigate random and systematic errors. The idea is that we will compare the time interval from the large digital "clock" at the front of the room, with a more precise instrument (the hand timers). We want to perform a test to see if the time scale of the digital clock is correct or not—that is, whether using the large digital clock would cause systematic errors were we to use it to measure time.

Although the timers have very accurate time scales, we need to use rather imprecise hand-eye coordination to operate them. The systematic error of the timers is small, and guaranteed by the manufacturer. But you will have to measure the random error from your hand-eye coordination, since at the start you don't know it. We will do it by repeating measurements of the same time interval on the large digital clock, and using the variation of the measurements to calculate the random error in a single time measurement. We will then use the fact that by repeatedly measuring time intervals, we can decrease the uncertainty of our estimate of the wall clock counting rate. So we are using an understanding of random errors to measure an effect (clock count rate) which might be a systematic error if used in another experiment without correcting for it.

4.2 Time Measurements

At the front of the room is a large digital clock. Assume that it counts at a constant rate, but do not assume that the rate is one count per second. The clock will count from 1 to 20, blank out for some unspecified length of time, and then begin counting again. To measure the clock's count rate, you will be given a timer whose systematic error is less than 0.001 seconds for time intervals of ten seconds or less. However, you will be relying on hand-eye coordination, which means your measurements will have random

uncertainties. Your reaction time is unavoidably variable. You may also be systematically underestimating or overestimating the total time.

4.2.1 Observe the clock. Write down whether you believe the clock is counting reasonably close to one count per second. Also, before doing any measurements, guess how much your reaction time would vary from one measurement to the next (this is your initial guess for your random error).

4.2.2 Now choose a counting interval at least 10 counts long, and time 25 of them. Avoid starting a timing interval on the first count. Use the first three or more counts to develop a tempo with which to synchronize your start. Write down in your notebook the initial and final clock count you use to define the interval.

One person should time while the other records the data on the data sheet belonging to the person timing. To avoid an unconscious skewing of data, the person timing should not look at the data sheet until all 25 measurements have been recorded. This is essential; otherwise, you will introduce a bias into your measuring procedure! Make a few practice runs before taking data. Exchange places with your partner, and time 25 more counting intervals. Each person will have a data sheet with 25 timings recorded on it. Assuming you have prepared well for the lab, you should both be able to analyze your own data. In any case, you should provide your own answers to the questions at the end.

5. Data Analysis

5.1 Plot your data first!   Then if it is wildly non-Gaussian, make another trial before sinking a lot of analysis time. Use *Kgraph* to make a histogram from your twenty-five measurements. The x axis represents the time measured T, and the y axis the number of measurements falling in the kth time bin. The data should be mainly peaked at a center value, and roughly symmetrical. You can consult with your instructor to see whether to take more data.

To analyze your data, you will calculate 3 quantities, the mean, the standard deviation, and the standard deviation of the mean value. If the formulas don't make sense to you, check Taylor chapter 4 again. The average or "mean" time per interval, $\overline{T}$ (pronounced "T bar"), is

$$\overline{T} = \frac{1}{N}\sum_{i=1}^{N} T_i$$

(1)

where the Σ stands for a summation, T_i represent the *i*-th single measurement of the time per interval, and N = the number of measurements. This formula directs you to add up the N values of T, and divide by N. Next you can use this value of $\overline{T}$ to compute the standard deviation, σ, of the values of T, defined as:

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(T_i - \overline{T}\right)^2}$$

(2)

These two quantities, $\overline{T}$ and σ, are characteristics of the distribution of your time measurements: the "center" (mean) and the "width" (standard deviation) of the distribution. As you make more measurements, your estimates should stabilize and converge to constant "true" values, though since each individual measurement you make varies randomly about the true values, your estimates will also vary, though less with more data available.

Finally, you can estimate the standard deviation of the mean. It is related to the standard deviation of the individual values:

$$\sigma_m = \frac{\sigma}{\sqrt{N}}$$
(3)

The standard deviation of the mean σ_m, is the best estimate of the *uncertainty* in the *measurement* of the mean. Note that, unlike the standard deviation, this uncertainty can be made arbitrarily small by taking a sufficiently large number of measurements.

5.2 To demonstrate that you understand them, write out the calculations of these three quantities explicitly (say with *Excel* but not using *Excel* statistical formulas) for the first 3 measurements (N=3), as if these were the only data you had taken. You may then use the computer for the full 25 values. Write in your notebook which item from the output of *Kgraph* Functions | Statistics does this calculation for you (How could you check that you are looking at the right entry? *Hint*: try N=3 first before N=25). You will need the values of σ and σ_m in what follows.

5.3 Now **predict** how $\overline{T}$, σ and σ_m vary with N, the number of measurements involved in their calculation. Then use *Kgraph* to calculate them for the first N=3, 5, 10, and 25 (all) of your measurements, again imagining that you had only the first 3, 5, 10, or 25 data points. Then comment whether on your predictions for the changes as N gets larger are approximately correct. In particular, why do σ and σ_m behave differently?

5.4 Compare your N=25 value of σ with your previous guess of the variability of your reaction time.

5.5 Now use *Kgraph* to make your final histogram from your twenty-five measurements. Adjust settings so the bin width to a "round" value about w ≈ 0.4σ Show the calculation in your notebook. How can you set the bin width *Kgraph* is using?

5.6 Clearly mark the points of $\overline{T}$ and $\overline{T} \pm \sigma$ for your measurements. These quantities can be shown to be the best estimate of your measurement.   The region included in the range $\pm\sigma$ should contain about 68% of your data points if your errors are random and consequently the distribution of your measurements is normal or Gaussian (Taylor, chapter 5). What fraction of your data lies within this range?

5.7 Extra Credit: (Taylor Chapter 5.2 – 5.3) Draw on your histogram an appropriate Gaussian distribution given by a curve (function) of the form

$$g(T) = Ae^{-\frac{1}{2}\left(\frac{T-\bar{T}}{\sigma}\right)^2}$$
(4)

To do this, you need values for the constants in the function g(T). For $\bar{T}$ and σ, use your best estimates for the mean and standard deviation of your time measurements. Choose A to match your histogram. *Hint*: what is the value of the function g(T) at T=$\bar{T}$?

Calculate g(T) at five points. (*Hint*: *Kgraph* and *Excel* both use *exp* for the exponential function). Draw the points by hand on your histogram plot. Then connect these points with a smooth curve, which should resemble the Gaussian curves in Taylor. With a finite number of measurements such as 25, your histogram may not resemble the expected "bell" shape curve to a great degree.

6. Drawing Conclusions from the Timing Measurements

6.1 Check to see is there a **(statistically) significant discrepancy** in the time measured by the large clock. We want to check the hypothesis that the large digital clock is running correctly. Statistical significance is tested for not by just looking at the size of the difference and saying "that seems small" or "looks big to me". Rather we compare the relative size of the difference with the uncertainty of our measurement. If the difference isn't substantially larger than our uncertainty, then we say that our statistical analysis left us with information insufficient to reject our original hypothesis, and we say that the differences are not statistically significant. We make the test quantitative by calculating the size of the discrepancy from expectations in units of the uncertainty of that difference.

The discrepancy we want is that between the average time counted, $\bar{T}$, with its expected value T_{exp} = (actual count/s) × (the number of counts in your interval). Your instructor will give you the actual counts/sec which was set before the start of the experiment.

Following the **uncertainties summary** in the Reference Guide, we use D = $\bar{T}$ - T_{exp}. We use σ_m for δD, since that is our uncertainty in how well we know $\bar{T}$, and there is no uncertainty in our prediction, T_{exp} . Then

$$t = \frac{\left|\bar{T} - T_{exp}\right|}{\sigma_m}$$
(5)

This expression is the same as Eq. 5.67 on p. 150 of your text, and one of the boxed equations in the summary on uncertainties. Here, in accord with standard statistical notation, *t* has the meaning of the number of the standard deviations of the mean needed to cover the difference between the mean and expected times. **In this equation, *t* is not a time**: it has no units since both the numerator and denominator are in seconds. The statistic t is handy because we can make such a comparison for any measurement: the numerator D and denominator δD always have the same units, so *t* is always means "number of standard deviations", a pure number on the same (dimensionless) scale, no matter what quantity was originally being measured. Following the handout, and

common statistical practice, we will say the deviation is **statistically significant if |t|>2**, that is the discrepancy is more than twice as big as our uncertainty.

As you will see later, this should happen by chance only about 5% of the time. Notice also if we make a sloppy measurement with a huge uncertainty δD, it will be hard to ever learn anything: it would be rare to have a large enough discrepancy to find a statistically significant difference from our starting assumption.

Based on this calculation, is $\overline{T}$ compatible with T_{exp} ? That is, is the discrepancy (statistically) significant?

6.3 Calculate the time interval per clock tick (How is this related to $\overline{T}$?). Does it appear that the clock tick is 1.0 seconds, as we assumed at the beginning? By how many % does your time differ from 1.0 seconds? What is the uncertainty in your estimate?

6.4 Suppose you had recorded only your first 3 measurements. What would you have concluded about the existence of a significant discrepancy? Were the remaining measurements necessary in your opinion? Give a quantitative justification.

6.5 Suppose you measured with the hand timer a different counter with a known period, and found your measured period was (statistically significantly) too low. What kind of flaw in your measurement procedures might have caused this bias?

6.6 What was the muddiest point of this experiment? Where, specifically, should the write-up be improved?