

## Experiment 6

### Simple Harmonic Motion

#### Reading:

Bauer&Westfall Chapter 13 as needed (e.g. definitions of oscillation variables)

#### Homework 6: turn in the second week of this experiment.

1. Define and explain briefly the meaning of the terms (a) restoring force, (b) free oscillation, (c) simple harmonic motion, (d) phase angle and (e) natural frequency.

Using Figure 2 in the write-up below,

2. What is the amplitude of the oscillation at  $t = 10 \text{ sec}$ ? at  $t = 0 \text{ sec}$ ?
3. Determine the period  $T$ , the angular frequency  $\omega$  and the phase angle  $\phi$  in  $\cos(\omega t + \phi)$ .
4. Redraw the oscillations in Fig. 2 for phase angle  $\phi = \pi/2$ .

#### Extra Credit:

5. Show by using Hooke's Law that a stationary mass  $m$  hanging from a spring with constant  $k$  (Fig. 1) stretches the spring to a new equilibrium position  $x_0 = \frac{mg}{k}$ .
6. Confirm by substitution that Eq. (7) indeed solves Eq. (6).
7. Show that when  $\beta = 0$  (no damping), Eq. (7) reduces to Eq. (3).

#### 1. Goals

1. Understand the properties of an oscillating system governed by Hooke's Law.
2. Study the effects of friction on an oscillating system, which leads to damping.
3. Use a non-linear least-squares fitting procedure to characterize an oscillator.
4. Learn the basics of using an oscilloscope.
5. Understand how to detect a power law relationship between variables.

#### 2. Theoretical Introduction

**2.1 Simple Harmonic Oscillation (SHO)** Consider a system illustrated in the figure below. It consists of a mass  $m$  suspended from a spring with spring constant  $k$ .

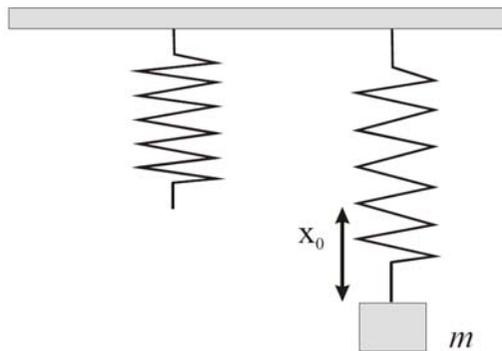


Fig. 1. A mass on a spring in the gravitational field of Earth

Hooke's law states that the force resisting the extension of the spring is proportional to the deviation of the spring from its equilibrium position. That is,  $F = -kx$ , where  $x = 0$  is defined

by the equilibrium position of the spring. If we gently add a mass to the spring, the spring will stretch to a new (lower) equilibrium position,  $-x_o = -mg/k$ , where  $g$  is the gravitational constant. At this position, the vertical restoring force of the spring balances the weight. In what follows, we will take the origin of  $x$  at this new equilibrium position. In these final coordinates, at  $x=0$  the gravitational force is canceled by the force due the spring, so that we can ignore the constant gravitational force in this coordinate system, and consider only additional Hooke's law forces beyond those required to cancel the weight.

If the mass is now displaced from its equilibrium position, the same equation  $F = -kx$  still applies, where  $x$  is now that displacement. In other words, when such a displacement is made, a restoring force again acts to return the mass to its equilibrium position. Upon release, the mass moves toward the equilibrium position, but its inertia causes it to "overshoot" this point. The motion then continues through the equilibrium position and beyond until the restoring force eventually stops the mass and pulls it back toward the equilibrium position. The motion then repeats itself back and forth through the position of equilibrium. Newton's second law states that any unbalanced force results in an acceleration of the mass, proportional to the force. If we apply Newton's second law to the motion of a mass  $m$  that is subject to Hooke's law, we get

$$m \frac{d^2 x}{dt^2} = -kx, \quad (1)$$

which can be written as

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = 0, \quad \text{where } \omega_0^2 = \frac{k}{m} \quad (2)$$

We have studied the solution of this equation in Experiment 3. The simple pendulum underwent simple harmonic oscillation with angular frequency  $\omega = \sqrt{g/L}$ , where  $g$  is the gravitational constant and  $L$  is the effective length. We found that the solution for the simple pendulum was expressed in a sinusoidal form,  $\Theta(t) = \Theta_o \sin(\omega t)$ . Therefore, we anticipate that the solution for Eq. (2) should be similar. However, we can't guarantee synchronization of the phase with passage through equilibrium, so leave it as a free parameter:

$$x(t) = A \cos(\omega_0 t + \varphi), \quad (3)$$

where  $A$  is the amplitude of oscillation,  $\varphi$  is a phase constant, and  $\omega_0$  is the angular frequency:

$$\omega_0 = \sqrt{\frac{k}{m}}. \quad (4)$$

**2.2 Damped Harmonic Oscillation (DHO)** The amplitude of oscillation of the mass gradually decreases over time. This is due to the effect of friction or a drag (resistive) force. We want to see if we can understand its effects. For our example, the effect of friction can be represented as a force proportional to the velocity of the mass. Therefore, Eq. (1) can be modified to read

$$m \frac{d^2 x}{dt^2} = -kx - \beta \frac{dx}{dt} \quad (5)$$

where  $\beta$  is a constant of proportionality, called the damping coefficient. The minus sign indicates that the damping force is always opposite to the direction of motion. Rearranging the above equation yields

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0, \quad (6)$$

where  $\gamma = \frac{\beta}{2m}$ . The solution to this (second order differential) equation is no longer SHO. If

$\beta$  is not too large, it is a modification of SHO. Also, the frequency of oscillation will be modified by the damping. The solution may be made plausible by an educated guess. We postulate that the oscillatory motion is still sinusoidal, but that the amplitude of oscillatory motion decreases as a function of time. So our trial solution is

$$x(t) = A e^{-\gamma t} \cos(\omega t + \varphi), \quad (7)$$

where  $A$  is the amplitude,  $\varphi$  is the phase angle, and it turns out that  $\omega = \sqrt{\omega_0^2 - \gamma^2}$  is the angular frequency of this system. See Appendix A for a more complete derivation. The coordinate  $x(t)$ , as a function of time  $t$ , is shown in Fig. 2.

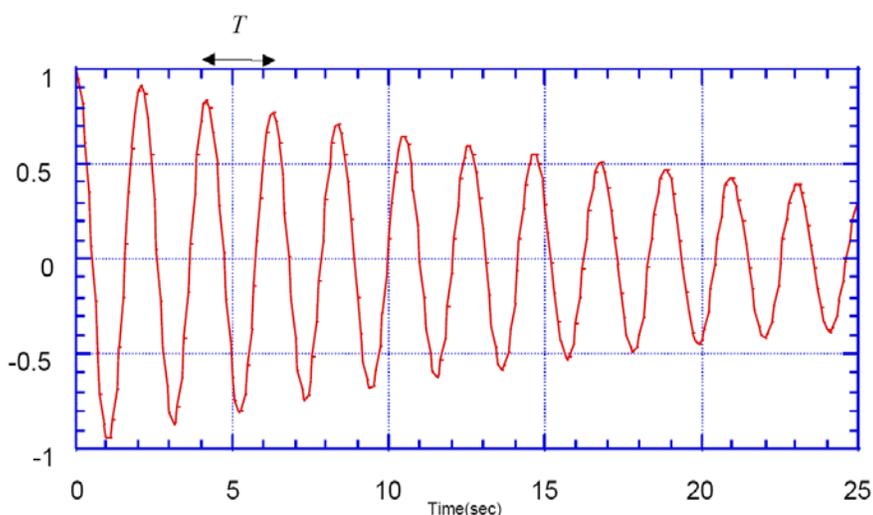


Fig. 2. Damped harmonic oscillator displacement as a function of time. The envelope decay function is  $\exp(-\gamma t)$ . The period  $T$  is related to  $\omega$  by  $T = 2\pi / \omega$ , where  $\omega = 2\pi f$ .

## Experimental Procedure

**3.1 The Dynamic Force Transducer** In this experiment we will use a Dynamic Force Transducer (DFT), an electronic device that outputs a voltage proportional to a force applied to it. In **Parts 7 and 8**, we will hook the outputs of the DFT to the input terminal block of a LabView card and use LabView to monitor the applied force. If we use the DFT as the spring support, then the force exerted by the spring is measured. If we ignore the mass of the spring (a constant), this force is the same as that applied to the hanging mass and is by Hooke's Law proportional to the displacement of the mass from equilibrium. Thus, by monitoring the force in the spring we are, in effect, monitoring the position of the mass. The transducer produces a voltage proportional to force, with a scale controlled by the "sensitivity" knob, so the scale is unknown unless we calibrate it. Further, we may use the "zero adjust" knob to zero the force readout for some one mass, but we use several different masses, and the zeroing may be inexact. Thus, we may regard the **DFT readout as giving not  $x$ , but  $Bx + C$** , and we should **add an arbitrary constant to any Kgraph equations used to fit DFT output**.

**3.2 Inertial Mass.** Throughout the lab, **whenever measuring mass, be sure to include the support hook** as well. (We will ignore the mass of the spring.). The mass in Eq. (1) is the *whole* mass that is accelerating—the inertial mass of  $F=ma$ . That of course is the same mass as enters in Eq 4 for the angular frequency.

**4. Checking DFT output vs. position.** There are two knobs on the DFT, zero adjust (or “offset”) and sensitivity (of “gain”). Set the sensitivity knob nearly fully clockwise and *do not touch it again during the experiment*. Measure the hook mass, then attach the spring and hook to the DFT. The output of the DFT goes to the **digital oscilloscope (described Appendix B: read before class!)**. With the spring and hook mass at equilibrium, set up the scope as described in the appendix, and then adjust the offset of the DFT until the output reads zero volt(s) on the digital scope display. Note that zeroing the output at equilibrium is for convenience only, but the resulting voltage **is** your measurement for the hook alone, with no mass added.

- 4.1. Measure the output (volts) for three more masses. The recommended order is 100g, 150g, and 200g (total of hook + added mass). Assuming Hooke’s law to be true (and the DFT to behave as advertised), write down the general equation relating output to the mass.
- 4.2. For these four measurements make a graph of  $V$  (volts) vs.  $m$  (grams). What function describes the dependence of  $V$  on  $m$ ? Is it roughly consistent with what you predicted? Do you think the DFT will do a good job of measuring position?

**5. Static Measurement of the Force Constant.** In this part you check Hooke’s Law by measuring the displacement vs. mass using a metric ruler.

- 5.1. Measure displacements for three (3) different masses. Assign uncertainties to the measured displacements; you can ignore the uncertainty of the masses. **Don’t forget to include the mass of the hook.**
- 5.2. Plot displacements (in units of cm) vs. mass (in units of grams) using *K-graph*.
- 5.3. Apply a least-squares fit procedure to obtain the spring constant  $k$  and its uncertainty. *Hint:* What are the units of  $k$ ? of the slope?
- 5.4. Predict the oscillation frequency  $\omega_0$  for each of these masses. Show the calculation in your notebook.

**6. Measurement of the Period of SHO.** In this part of the experiment you will use the oscilloscope T cursors to measure (for 4 mass values) the angular frequency of oscillation  $\omega_0$ .

- 6.1. Attach your first mass to the spring and set it into oscillation. Pick an appropriate point of reference and measure the **time** for several complete cycles of the oscillation. Record the time in your notebook (and lab report). What’s more accurate: a single cycle, or several? Why?
- 6.2. Repeat step 1 for your second and third mass you used in Part 5.
- 6.3. To obtain  $\omega_0$ , note that  $\omega_0 = 2\pi f$ , where  $f$  is measured in cycles per second.
- 6.4. Compare these  $\omega_0$  with those you predicted in Part 5.
- 6.5. Now do *one more* mass value you haven’t done before.
- 6.6. Using *K-graph*, plot your data of  $\omega_0$  vs.  $m$ . Now, using the RG note on Log Plots, make another plot with one or more axes logarithmic, which tests whether you observe the functional dependence predicted by Eq. (4). *Hint:* what shape, and what slope, would you observe if Eq.(4) were true?

**7. Oscillating Spring: Displacement as a Function of Time.**

- 7.1. Attach the spring-hook-mass (150g total) combination to the DFT. With the help of the digital scope, adjust the DFT offset so the signal is symmetric about 0V. Set the spring-mass system into oscillation, observe the pattern on the scope display and sketch it in your notebook (and lab report).
- 7.2. The output of the DFT is also connected to a data acquisition card on the PC. Open R:\exp7\_ref\ForceTransducer.vi. The following specifications are needed in order to collect the data successfully. Set **device** = 1; **channels** = 0; **scan numbers** = 1000, **scan rate** = 100. The **scan numbers** is just the number of data points; the **scan rate** is the number of points/second, so the number of seconds for which you will record data is just **scan numbers/ scan rate**. If you need a longer data sample, raise the scan numbers; if you need finer sampling, raise the scan rate. Note: if you are using a flash drive, be careful to NOT remove the USB DAQ device along with your flash drive.

Set the spring-mass system into oscillation and **click the arrow button** or press **Ctrl R** to begin collecting the data. (The scanning process will take 10 seconds.) When it is done, the program will ask you to save the data on the U drive. **Caution:** When exiting the program, it will ask you if you want to save the current setting. Choose **NO**.

- 7.3. Use *K-graph* to retrieve the data file. Use “Any files \*.\*”, not “All Files”; don’t ask me why, but they aren’t equivalent. When inputting your data to *Kgraph*, the following specifications are required: **Delimiter** = Tab; **Number** = 1, **Line Skipped** = 0, **Options/Read Title** (No check). Your file contains two columns: the first column is the time, the second column is the output voltage. **The data is big: don’t print data tables**.
- 7.4. Plot voltage vs. time and use the general curve-fit editor to perform a non-linear least squares fit procedure using the function given in Eq. (3). The corresponding *Kgraph* function is  $m1+m2*\cos(m3*x + m4)$ ;  $m3$  and  $m4$  are in radians ( $m1$  is the unknown voltage for  $x=0$ ). Explain why you need the  $m1$  parameter, since it’s not in Eq (3).

Notes:

- This non-linear least-squares fit function will not converge unless your initial parameters ( $m1 \dots m4$ ) are realistic—especially for the angular frequency, for which you must find a way to get a good estimate. **Show in your notebook how you estimated each parameter**. How do the values compare to those output by *K-graph*?
- The fit is subject to the ambiguities inherent in trig functions. The parameters of the fit should be reduced to standard form using trig identities. You should wind up with a positive amplitude, a positive  $\omega$ , and a phase angle in the range  $[-\pi, \pi]$ . A simple procedure is to first correct the sign of the amplitude (if necessary) by  $-\cos(z) = \cos(z+\pi)$ , then correct the sign of  $\omega$  if necessary by  $\cos(-z) = \cos(z)$ , and finally correct  $\phi$  by  $\cos(z) = \cos(z \pm 2N\pi)$ , where  $N$  is any integer. Show the steps in your notebook, and report the final values in your report or hand-written on your plot (which should show the original fit values and uncertainties)

## 8. The Effect of Friction

In this part of the experiment, we will investigate the behavior of the spring oscillations under the effect of friction.

- 8.1. Weigh the friction umbrella. Attach the friction umbrella to your spring-hook-mass (~150g) combination and once again observe the spring tension on the digital scope. Adjust the **offset** on the DFT if necessary. Set the system into oscillation and observe the oscillation pattern on your digital scope. You should observe a pattern similar to that in Fig. 2, which shows the sinusoidal oscillation modulation by the exponential damping factor.
- 8.2. Measure the period  $T$  on the digital scope using the built in buttons. Calculate the angular frequency  $\omega$  and the frequency  $f$  from  $T$  and record these values in your notebook.
- 8.3. With the friction umbrella attached to your spring-mass system as in step 1, repeat steps 2 and 3 of Part 7 (you may need to increase to 2000 data points). Fit the data to the function given by Eq. (7):  $m_1 + m_2 \exp(-m_3 x) * \cos(m_4 x + m_5)$ . Show how your  $m_1 \dots m_5$  parameters are estimated. How do these values compare to those given by *K-graph*? As above, standardize the amplitude, angular frequency, and phase. *Hint*: if you're stumped on  $m_3$ , see 9.2 below.
- 8.4. Repeat all of the steps 8.1 to 8.3 (measurements and fits) with a 100g mass total.

## 9. Questions and Analysis

- 9.1. In Part 5 we calculated the spring constant  $k$  from the slope of a plot of displacement vs. mass. Using the same data, give an alternative method for making a plot that yields the spring constant  $k$  directly from the slope. *Hint*: What kind of plot would have a slope that is the spring constant  $k$ ? Alternative: substitute constants and arrange the fit constant so the fit parameter is  $k$  itself.
- 9.2. In this question, you are asked to estimate the decay factor directly from a plot of voltage vs. time. Draw a smooth curve connecting the decay peaks on your plot in Part 8. The envelope decays as  $\exp(-\gamma t)$ . When  $t = \tau \equiv 1/\gamma$ , the amplitude  $x(t)$  has decreased by a factor of  $1/e$ , or about 0.368 times the initial value  $A$ . Beginning from any point on the time-axis, determine  $\tau$ , the length of time required for the amplitude  $x(t)$  to decrease by  $1/e$ . Compare the resulting  $\gamma$  with that given by *K-graph fit*. *Hint*: If you didn't cover a long enough time to reduce by .368, think about the decrease you'd get in a time  $\tau/2$ .
- 9.3. Does  $\gamma$  from Part 8 depend on the mass of the system? Calculate  $\gamma$  supposing you had 200g total, and compare it to the values for mass of 100g and 150g.
- 9.4. **Extra Credit**: What is the % difference between your angular frequency of free oscillation  $\omega_0$  and your prediction in Part 5? A more detailed theory suggests adding  $M_{\text{spring}}/3$  to the suspended mass when calculating the frequency. The spring weighs about 9 grams. Does this fit your observed  $\omega_0$  better than just using the suspended mass? Is either prediction compatible with your measurement with uncertainties?
- 9.5. **Extra Credit**: Compare the angular frequency of free oscillation  $\omega_0$  (found in Part 6) with the damped oscillator frequency  $\omega$ . Again use data with  $m = 150\text{g}$ . Discuss your comparison of  $\omega$  for the damped oscillator with  $\omega_0$  for free oscillator. Is the effect of the mass of the friction umbrella on  $\omega_0$  significant in this comparison? Are your results roughly consistent with Eq A5 for  $\omega$  and the value of  $\gamma$  from your fit? Which contributes more: the mass of the umbrella, or the correction for  $\gamma$ ?
- 9.6. Summarize your measurements in two tables, one for angular frequencies  $\omega$  and  $\omega_0$ , and another (shorter) for estimates of  $\gamma$ . The  $\omega$  table should include the method, mass, value, prediction, and difference in %; base the predictions on the results from Part 5.

## Appendix A: Solution of Damped Harmonic Oscillator

The solution Eq. (7) for the damped harmonic oscillator in Eq. (6) can be found as follows.

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0. \quad (\text{A1})$$

The general solution is

$$x(t) = Ae^{Qt}, \quad (\text{A2})$$

where  $Q$  is a factor to be determined. Then

$$\frac{d}{dt}x(t) = QAe^{Qt} = Q \cdot x(t) \quad \text{and} \quad \frac{d^2}{dt^2}x(t) = Q^2 Ae^{Qt} = Q^2 \cdot x(t)$$

Substitution of these two in (A1) yields

$$(Q^2 + 2\gamma Q + \omega_0^2)x(t) = 0, \quad (\text{A3})$$

which has a solution of the form

$$Q = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}. \quad (\text{A4})$$

For our case, which involves a weakly damped oscillation,  $\omega_0 \gg \gamma$ . Therefore, our solution (A4) becomes

$$Q = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2} = -\gamma \pm i\omega \quad \text{with} \quad \omega = \sqrt{\omega_0^2 - \gamma^2} \quad \text{and} \quad \omega_0^2 = k/m \quad (\text{A5})$$

Our general solution Eq. (A2) now becomes

$$x(t) = A_1 e^{-\gamma t + i\omega t} + A_2 e^{-\gamma t - i\omega t}. \quad (\text{A6})$$

Set  $A_1 = \frac{1}{2} A e^{i\varphi}$  and  $A_2 = \frac{1}{2} A e^{-i\varphi}$ , where  $\varphi$  is a phase factor. Then

$$x(t) = \frac{A}{2} e^{-\gamma t} e^{i(\omega t + \varphi)} + \frac{A}{2} e^{-\gamma t} e^{-i(\omega t + \varphi)} = A e^{-\gamma t} \frac{1}{2} (e^{i(\omega t + \varphi)} + e^{-i(\omega t + \varphi)}). \quad (\text{A7})$$

But  $\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$ . Thus,

$$x(t) = A e^{-\gamma t} \frac{1}{2} (e^{i(\omega t + \varphi)} + e^{-i(\omega t + \varphi)}) = A e^{-\gamma t} \cos(\omega t + \varphi), \quad (\text{A8})$$

which is the solution Eq. (7).

## Appendix B: Oscilloscope hints

An oscilloscope (“scope”) (photo on last page of write-up) is a device to perform real-time visualization of voltages in an electric circuit. The screen of the scope shows the dependence of the measured voltage on time. If the real voltage changes, the picture on the screen changes as well. The time is shown on the x-axis, the voltage on the y-axis. Just as in *KGraph*, you can choose the best presentation of the graph by adjusting various scope controls. And just as with *Kgraph* presentation settings, **no matter what scope settings you change, you have not changed anything about what happens in the circuit.** If you know how many seconds (volts) correspond to one tick on the grid of the screen, you can read the period and the magnitude of the signal directly off the screen, or more accurately by using the cursor buttons.

### Setup of the oscilloscope

Your scope can simultaneously measure two voltages, but in this lab you need only one input signal (supplied through channel A). The current settings for channel A (B) are shown in the upper part of the digital display to the right of the screen. By pressing the “A/B” button make sure that the shown settings correspond to channel A. The only settings you should have to change on the scope are *circled in the photo*: those on the top row, the TB setting at the middle towards the left, and the row second from the bottom. Leave alone the VAR knob: it should point to CAL (all the way to the right), and leave alone all the channel B settings.

Connect a BNC cable from the transducer to the channel A input. Turn on the scope. Press “AC/DC” button until the digital display shows “DC”. (“DC” shows the real voltage; “AC” only shows variations about the average). Also, turn the “DIGITAL MEMORY” on. Choose the time base (the value of the 1 cm grid of the time scale) with the “TB s  $\mu$ s” switch to be equal to “0.5 s”, and the voltage scale with the “A V – mV” switch to be equal to “0.1 V”. During the lab, you may want to adjust these settings to get the best presentation of the graph. It’s also nice to push XMAGN 6 times or so until the display shows [-----] instead of [---- ] so the scope continuously collects new data. The TB will now be 1.0s; re-adjust it to .4s .

Press “GND”. This, connects the scope input will be to “ground” (= 0V). Turn the Y-pos and X-pos knobs to position the green line of the signal exactly in the middle of the screen. Do not touch the X- pos and Y-pos buttons for the rest of your measurements. This ensures the correct calibration of the picture offsets inside the scope. Then you will only need to worry about the voltage offset inside the force transducer, which you adjust with “ZERO” knob of the transducer.

Press “GND” again. Now you will see the signal coming from the transducer. For nonzero mass, it will be vertically shifted with respect to the center of the screen. For static stretching of the spring, you will measure this displacement of the signal to find the dependence between the applied mass, elongation of the string and the voltage. During the study of oscillations, you will have to adjust the “ZERO” on the force transducer until the signal line is again in the middle. Thus you will make sure that the voltage shown corresponds to the displacement of the mass from the equilibrium point, and not from some other location of the system.

Several **other useful buttons**: you may use the knobs to the left of the screen to adjust the focus and the brightness of the line. The button “LOCK” will allow you to lock (freeze) the image, which makes detailed measurements on the captured image convenient, as described below.

## Voltage and Time Measurements with the Cursors

Inspect the main screen. If there is no writing at the bottom of the screen, press one of the blue "soft-keys below the screen to make the writing appear. If there is some writing and one of the soft-keys has RETURN written above it, keep pressing it until it no longer says RETURN, to get back to the top menu. You should see:

CURSORS                      SETTINGS                      TEXT OFF

**Reading Voltages.** We will use these soft-keys to move screen cursors so they correspond to the size of our signal and read off voltages from the screen. Press the CURSORS soft-key. Press MODE to set up the cursors we want. Toggle the V-CURS and T-CURS soft-keys until the horizontal cursor lines are on and the vertical cursor lines are off. Press RETURN.

Press V-CTL to control the Voltage, cursors. The V cursors measure voltage, which is displayed as the vertical (y) axis on the screen. They appear as horizontal lines on the screen, representing a constant voltage level. Move the top one down to the level of your signal, and move the bottom one up to the reference value. The REF line must be lower than the  $\Delta$  line, so **if you are measuring a negative voltage**, you'll have to add the sign. The value is displayed at the top of the screen. For uncertainty estimation: use 2 clicks, or two line widths, or re-measure a few times.

**Frequency Measurement:** Next, we wish to determine the time between 2 points. Time is measured as a horizontal distance on the screen. The sine-wave signal is cyclic: it repeats itself. The time a sine-wave signal takes to make one cycle is called the period of the signal. It has units of seconds. The inverse of the period is the frequency (in Hz).

To measure time we need the T-cursors (T stands for time). Time is displayed as the x-axis on the screen, so a constant time is marked by the T cursor as a vertical line. Return to the top-level menu using the RETURN soft-key. Press MODE and toggle the T-CURSOR to on. Hit RETURN, then hit T-CTL to move the time-cursors to the left and right. The length of the cycle can be measured by positioning the T-cursors on equivalent points on the trace, for example, on two adjacent peaks or two adjacent valleys. The period and the frequency can both be read off from the top of the screen, but the frequency is "calculated" rather coarsely, so **measure the period and calculate the frequency yourself by  $f = 1/T$  !**

