PHY321 Homework Set 10

1. [10 pts] Consider the central-force problem for a potential $U(r) = -k/r^2$ and a reduced mass μ. Here, k is a positive constant.

   (a) Sketch the effective potential for large and small values of angular momentum $|\ell|$. Use your graph and energy conservation and show that at large ℓ the orbits extend from some minimal value of radius $r_{\rm min}$ up to infinity. Obtain $r_{\rm min}$ as a function of energy E. Show that at small ℓ the orbits reach $r = 0$, but can either extend up to a finite $r_{\rm max}$ or up to infinity. What value of $|\ell| = \ell_s$ separates the low and high value regions of ℓ, with different types of orbits? Obtain $r_{\rm max}$ as a function of E.

   (b) Turn to the orbit equation and find general analytic solutions $r(\theta)$ for high and low values of $|\ell|$. (Note that, $\ell = \pm\ell_s$ requires a separate solution, but you are not asked to investigate it here.)

   (c) As an example, adjust the arbitrary constants in the solution $r(\theta)$ for *high* $|\ell|$, so that the orbit is symmetric about the axis $\theta = 0$ and that the orbit extends from $r_{\rm min}$ on. Note that at $r_{\rm min}$, $\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{r} = 0$. Sketch the orbit in the plane of motion. What are the asymptotic directions at $r \to \infty$, in terms of angle, for the orbit?

   (d) As another example, adjust the arbitrary constants in the solution $r(\theta)$ for low $|\ell|$, so that the orbit is symmetric about the axis $\theta = 0$ and that it extends up to $r_{\rm max}$. Again, at $r_{\rm max}$, $\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{r} = 0$. Sketch the trajectory in the plane of motion for $\theta > 0$.

2. [5 pts] Consider a particle of mass m moving under the influence of a central force

$$F = -\frac{k}{r^2} + \frac{c}{r^3}\,,$$

   where k and c are positive constants.

   (a) Demonstrate that the solution to the orbit equation can be put into the form

$$\frac{\alpha}{r} = 1 + \epsilon\cos(\nu\theta)\,,$$

   which is an ellipse for $\epsilon < 1$ and $\nu = 1$. How are ν and α related to c, k and ℓ? What happens to ν when c approaches zero?

   (b) Discuss the character of the orbit when $\epsilon < 1$ and $\nu \neq 1$. Hint: Consider the change in the angle after one period in r, e.g. when reaching again $r = r_{\rm min}$. Sketch the orbit assuming a small deviation of ν from 1. Note: Effects of general relativity contribute minute corrections to $F = -k/r^2$ for planets. Those corrections are characterized by a faster fall-off than $1/r^2$ and they cause modifications of the orbits compared to elliptical.

3. [10 pts] A satellite moves in an elliptical orbit around Earth.

   (a) If the ratio of the maximum angular velocity to the minimum angular velocity for the satellite is 3.4, what is the eccentricity ϵ of the orbit? Hint: Use angular momentum conservation.

   (b) If the perigee of the orbit is 300 km above the Earth's surface, how high is the apogee above the surface?

   (c) What are the semimajor and semiminor axes of the orbit?

   (d) What is the period of the orbit?

   (e) What are the linear and angular velocities of the satellite at the perigee and at the apogee? Hint: Start with aerial velocity.

   (f) If the satellite were to be slowed down so that it could reach the Earth's surface, would it be better to do it at the perigee or apogee? Calculate the required change in linear velocity Δv for both locations. Hint: Consider change in energy associated with change in semimajor axis.

   (g) If the exhaust velocity for satellite boosters is $u = 3100\,\mathrm{m/s}$, what fraction of the satellite mass would need to be burnt up as fuel to arrive at the desired change in velocity for the more convenient of the locations? For simplicity assume that the boosters operate over such a short time that effects of gravitational and centrifugal forces can be ignored.

4. [5 pts] Consider objects of mass m moving under the influence of a central gravitational force characterized by potential energy $U = -k/r$.

   (a) Demonstrate that when two orbits, circular and parabolic, have the same net angular momentum, the parabolic orbit has a perihelion that is half the radius of the circular orbit.

   (b) The speed of a particle at any point of a parabolic orbit is larger by a factor of $\sqrt{2}$ than the speed of particle on a circular orbit passing through the same point.

5. [5 pts] At perihelion, a particle of mass m in an elliptical orbit in a gravitational $1/r$ potential receives an impulse $m\,\Delta v$ in the radial direction. Find the semi-major axis a_{new} and eccentricity ϵ_{new} of the new elliptical orbit in terms of the old ellipse's parameters a_{old} and ϵ_{old}. Are these parameters increasing or decreasing in effect of the impulse. Does the direction of the impulse, inward or outward, matter for those specific parameters?