PHY321 Homework Set 6

1. [5 pts]

   (a) To solve the equation of a damped harmonic oscillator

   $$\ddot{x} + 2\,\beta\,\dot{x} + \omega_0^2\,x = 0\,.$$

   assume the solution to be of the form

   $$x(t) = f(t)\,\mathrm{e}^{-\beta t}\,,$$

   and obtain an equation for $f(t)$.

   (b) Obtain the solutions of the latter equation for $\omega_0 > \beta$ and $\omega_0 < \beta$. Show, further, that the general solution for $\omega_0 = \beta$ is $f(t) = A + B\,t$, where A and B are constants.

   (c) Write out the corresponding forms of $x(t)$ for the three cases of β/ω_0.

2. [10 pts]

   (a) Given $x(t) = A\,\mathrm{e}^{-\beta t}\cos(\omega_1 t - \delta)$ for an underdamped harmonic oscillator, with $\omega_1^2 = k/m - \beta^2$, obtain an expression for the time dependence of net energy, $E(t) = m\,\big(\dot{x}(t)\big)^2/2 + k\,\big(x(t)\big)^2/2$, with separated and organized smooth and oscillating contributions as a function of time. An example of the time dependence of the net energy is displayed in the top panel of Fig. 3-8 in the textbook.

   (b) Obtain an expression for the rate of change of energy of the damped harmonic oscillator, again with separated and organized smooth and oscillating contributions, by directly differentiating the energy from 2a. An example of the rate of energy loss is displayed in the bottom panel of Fig. 3-8 in the textbook.

   (c) Independently obtain the rate of energy change from the rate at which the resisting force carries out work, $\mathrm{d}E/\mathrm{d}t = \mathrm{d}W/\mathrm{d}t\big[\equiv P\big] = F_{\mathrm{res}}\,\dot{x} = -b\,\dot{x}^2 \le 0$. Do the rates obtained in the two different ways agree?

   (d) For a weakly damped harmonic oscillator with $\beta T_1 \ll 1$, find an approximate expression for the net energy averaged a period $\langle E\rangle(t) = (1/T_1)\int_t^{t+T_1} E(t')\,\mathrm{d}t'$.

   (e) For a weakly damped harmonic oscillator with $\beta T_1 \ll 1$, find an approximate expression for the rate of energy loss averaged over a period $\langle P\rangle(t)$.

   (f) Do the displacement amplitude and average energy decrease at the same rate? What is the difference if any?

3. [5 pts] Consider critically damped harmonic oscillator, for which

   $$x(t) = (A + B\,t)\,\mathrm{e}^{-\beta t}\,.$$

   (a) Express the constants A and B in terms of the initial values for the oscillator, $x(t = 0) = x_0$ and $\dot{x}(t = 0) = v_0$.

(b) Sketch four phase-space trajectories for this oscillator, starting in the four different quadrants in the phase space corresponding to four combinations of signs of x_0 and v_0.

(c) Prove analytically that a phase-space trajectory for the critically damped oscillator approaches the line $\dot{x} = -\beta\, x$ as $t \to \infty$.

(d) Are the phase-space trajectories for the critically damped oscillator circling around the origin in phase space?

4. [5 pts] Consider a nondriven overdamped oscillator for which $x(t=0) = x_0$ and $\dot{x}(t=0) = v_0$.

   (a) Within the solution of the equation for the damped harmonic oscillator, written as
   $$x(t) = A_1\, \mathrm{e}^{-(\beta+\omega_2)\, t} + A_2\, \mathrm{e}^{-(\beta-\omega_2)\, t}\,,$$
   where $\omega_2 = \sqrt{\beta^2 - \omega_0^2}$, express the coefficients A_1 and B in terms of x_0 and v_0.

   (b) Consider a strongly overdamped oscillator where $\beta \gg \omega_0$. What is a qualitative difference in the t-dependence between the two terms in $x(t)$ with coefficients A_1 and A_2? Which term will dominate the changes in $x(t)$ at short times and which at long times and why?

   (c) Explain the qualitative features of the phase diagram for the overdamped oscillator in Fig. 3-11 of the textbook. Why are the early parts of the paths parallel to the line $\dot{x} = -(\beta + \omega_2)\, x$ and the late parts follow the line $\dot{x} = (\beta - \omega_2)\, x$?

5. [5 pts] Consider *velocity resonance curve* for a driven damped harmonic oscillator characterized by ω_0 and β parameters.

   (a) From $x(t) = A\cos(\omega t - \delta)/\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\, \beta^2}$, obtain an amplitude for velocity oscillations in the driven motion. For what frequency ω is the velocity amplitude maximal and what is the value of the maximal velocity?

   (b) The width of a resonance curve $\Delta\omega$ is defined as the difference between the two frequencies at the two sides of the resonance peak where the velocity amplitude drops to $1/\sqrt{2}$ of the maximal. Find the width of the velocity resonance curve.

   (c) Draw the velocity resonance curve for the case of an oscillator with $Q = 5$.

6. [5 pts] Consider an oscillator consisting of a 0.20 kg mass attached to a spring with a force constant of 5.0 N/m and immersed in a fluid that supplies a damping force represented by bv with $b = 0.30\,\mathrm{kg/s}$.

   (a) What is the nature of the damping (over-, under- or critical)?

   (b) Next the oscillator is attached to an external driving force varying harmonically with time as $F_d(t) = F_0 \cos\omega t$, where $F_0 = 1.00\,\mathrm{N}$ and $\omega = 4.0\,\mathrm{rad/s}$. What is the amplitude of the resulting steady-state oscillations?

   (c) What is the resonant angular frequency of the system i.e. what is the frequency ω_R that the driving force has to be tuned to in order to produce a maximal amplitude?

   (d) What is that maximal amplitude?

   (e) What is the Q-value for this system?