PHY321 Homework Set 7

1. [10 pts] Consider a driven harmonic oscillator for which the equation of motion is

$$m\,\ddot{x} = -k\,x - b\,\dot{x} + F_0\,\cos\omega t\,.$$

   (a) Given the stationary solution to the equation

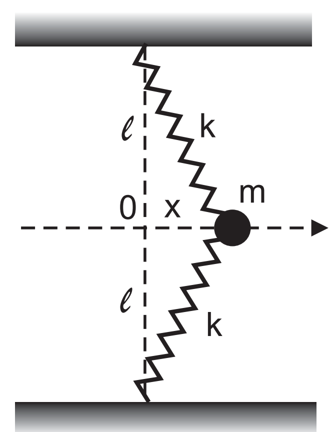
   $$x(t) = D\cos\left(\omega t - \delta\right),$$

   where D and δ are functions of ω, find the instantaneous $P_{\mathrm{drv}}(t) = F_{\mathrm{drv}}\,\dot{x}$ and average power $\langle P_{\mathrm{drv}}\rangle$ delivered by the driving force $F_{\mathrm{drv}} = F_0\,\cos\omega t$. Simplify the results as much as possible, in particular attempting to separate the oscillating from constant terms in $P_{\mathrm{res}}(t)$ and to illuminate the dependence on the phase shift δ.

   (b) Find the instantaneous $P_{\mathrm{res}}(t) = F_{\mathrm{res}}\,\dot{x}$ and average power $\langle P_{\mathrm{res}}\rangle$ lost to the resistance force $F_{\mathrm{res}} = -b\,\dot{x}$. Again, simplify the results as much as possible. Do the powers associated with the driving and resistance forces balance each other, whether instantaneously or on the average?

   (c) For what angular frequency ω is the average rate of energy transfer from driving to resistance force the maximal? What is the phase shift δ then?

2. [5 pts] Consider mass m attached by two identical springs, of spring constant k, to the rigid supports separated by distance of 2ℓ, as shown in the figure. The neutral length for each spring is $d < \ell$.



   (a) What is the potential energy $U(x)$ for this system? Make a sketch of $U(x)$, assuming $d \sim \ell/2$. Make any characteristic features of $U(x)$ explicit.

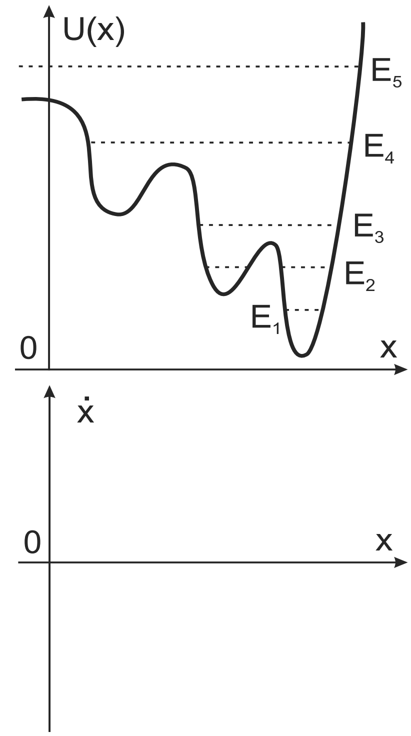
   (b) Expand the potential energy for small $x \ll \ell$, retaining one nonvanishing term beyond quadratic.

   (c) What kind of nonlinearity is present in this system? How is such a system termed?

   (d) Obtain a returning force from the expanded energy.

   (e) Sketch phase diagrams for 3 energies of progressively increasing magnitude, illustrating transition from linear to nonlinear regime for this system.

3. [5 pts] A particle of mass m moves in one dimension under the influence of a conservative force for which the potential energy $U(x)$ is shown in the figure. Sketch phase diagrams for the 5 energies of the particle, indicated in the figure. Mark which phase diagram is for which energy.



4. [5 pts] A plane pendulum consisting of a mass m attached to a massless rod of length ℓ starts from rest at an angle θ_0. Given that the pendulum spends quarter of its period moving between angles $\theta = 0$ and $\theta = \theta_0$, the period can be calculated from

$$\frac{T}{4} = \int_0^{T/4} \mathrm{d}t = \int_0^{\theta_0} \frac{\mathrm{d}t}{\mathrm{d}\theta}\,\mathrm{d}\theta\,.$$

(a) Use energy conservation to obtain $\mathrm{d}\theta/\mathrm{d}t$ as a function of θ for a given θ_0. Use this result under the integral above to arrive at an integral expression for the period T as a function of the maximal angle θ_0. Comment on the presence or absence of the dependence of T on m, ℓ and θ_0.

(b) Use a small-angle expansion in the subintegral function, to arrive at the period in the limit $\theta_0 \to 0$. Does your result agree with that from solving the equation of motion in the small-angle approximation? Systematic corrections to the latter period, in powers of θ_0, may be arrived at by employing Taylor-expansions in the subintegral function of the expression obtained in 4a. Useful integral: $\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x$.

5. [5 pts] Consider the mapping function $x_{n+1} = \alpha \sin(\pi x_n)$. Here, α is a constant characteristic for the map, x is restricted to the range $0 < x < 1$ and the argument of the sine is in radians.

(a) What is the asymptotic value of the mapping function when $\alpha = 0.6$? Depending on your starting value, the asymptotic value should be reached, to a very good approximation, in ~ 10 iterations or less.

(b) What are the *two* asymptotic values for the mapping function when $\alpha = 0.73$?