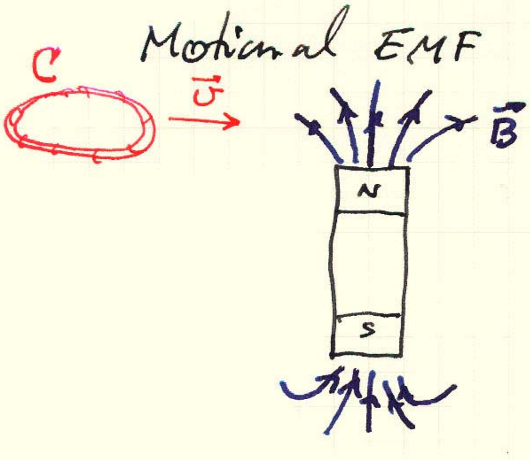
Electromagnetic Induction

— Faraday's Law

Motional EMF



The magnet is fixed in place; the circuit C moves to right. There is no electric field.

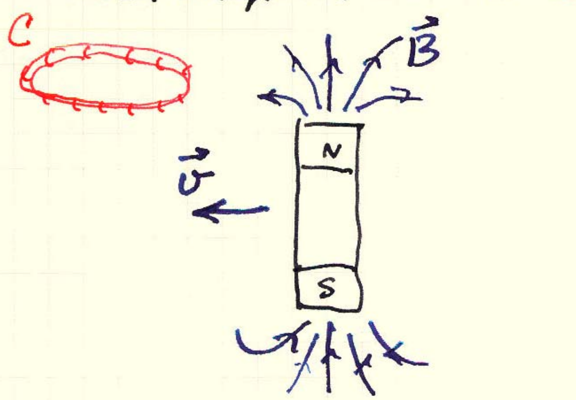
$$\text{EMF} = \oint_C (\vec{v} \times \vec{B}) \cdot d\vec{\ell}$$

$$= -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{A}$$

(proven last time)

Units $\frac{m}{s} \cdot T \cdot m = \frac{1}{s} Tm^2$ ✓

Electromagnetic Induction



The circuit C is fixed in place; the magnet moves to the left. IN THIS FRAME of REFERENCE THERE IS an INDUCED ELECTRIC FIELD.

$$\text{EMF} = \oint_C \vec{E} \cdot d\vec{\ell}$$

$$= -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{A}$$

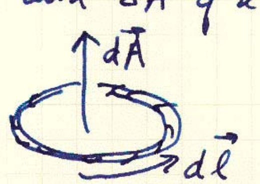
(Faraday's law)

Units : $\frac{V}{m} \cdot m = \frac{1}{s} Tm^2$

$1\ V/m = 1\ Tm/s$ ✓

(volt $= J/C$; tesla $= \frac{Vs}{m^2} = \frac{J}{Am^2}$)

Relation between directions $\vec{d\ell}$ and $d\vec{A}$ of a loop:



i.e., right hand rule

Faraday's Law

When $\vec{B}$ is changing in time, there exists an induced electric field $\vec{E}$.

For any fixed loop C,

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{A}$$

- Differential form of Faraday's law

By Stokes's theorem, $\oint_C \vec{E} \cdot d\vec{l} = \int_S \nabla \times \vec{E} \cdot d\vec{A}$

Thus $$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Note that electromagnetic induction is PURELY A FIELD EFFECT.

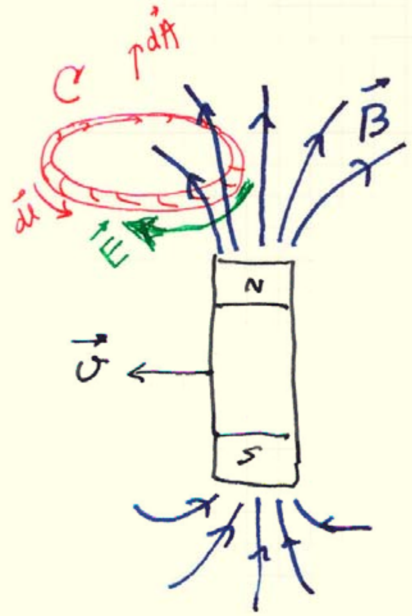
This phenomenon has important applications in technology:

— electric generators

— transformers

— inductance (e.g., in AC circuits)

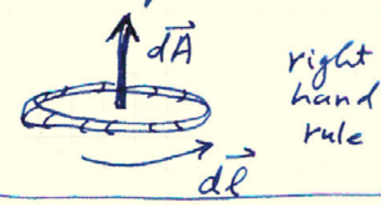
Lenz's Law



$$\nabla \times \vec{E} = -\partial \vec{B}/\partial t$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$

The relationship between the directions of $d\vec{l}$ and $d\vec{A}$



Consider the flux of $\vec{B}$, upward through C.

As the upward flux increases, the counterclockwise EMF around C is negative.

I.e., the induced electric field is clockwise.

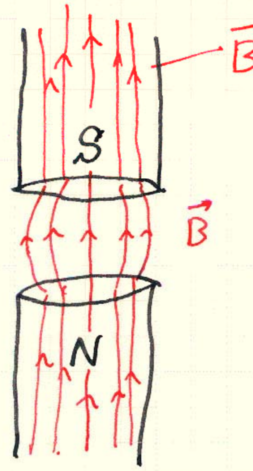
$$\underbrace{\int \vec{B} \cdot d\vec{A}}_{\text{upward}} \text{ in increasing so } \underbrace{\oint \vec{E} \cdot d\vec{l}}_{\text{counterclockwise}} \text{ is negative}$$

Lenz's law If a conductor is present at C, then the direction of the induced current opposes the change of magnetic flux.

{ I.e., the induced current creates a magnetic field (by Ampère's law) in the direction to maintain the magnetic flux. }

Example

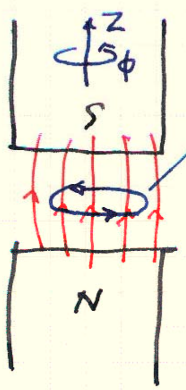
Consider a cylindrical electromagnet as the magnetic field increases, as shown.



$\vec{B}$ is approximately uniform in the gap between the poles,

$$\vec{B} = B(t)\,\hat{k}.$$

Determine the electric field.



loop C with radius r

$\vec{E}$ curls around the change of $\vec{B}$ in the direction given by Lenz's law.

Use cylindrical coordinates. (z, r, ϕ)

$$\vec{E} = E_\phi(r,t)\,\hat{\phi}$$

$$\oint_C \vec{E}\cdot d\vec{\ell} = E_\phi \cdot 2\pi r \quad \text{and} \quad -\int_S \vec{B}\cdot d\vec{A} = -\overline{B}\,\pi r^2$$

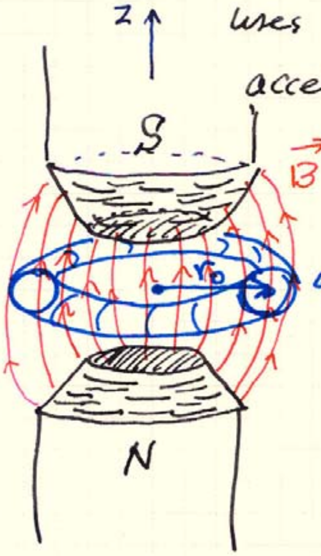
where $\overline{B}$ is the average of B_z over the interior (S) ; $\overline{B} = \frac{1}{\pi r^2}\int \vec{B}\cdot d\vec{A}$

Faraday's law $\Rightarrow$ $E_\phi \cdot 2\pi r = -\frac{d\overline{B}}{dt}\cdot \pi r^2$

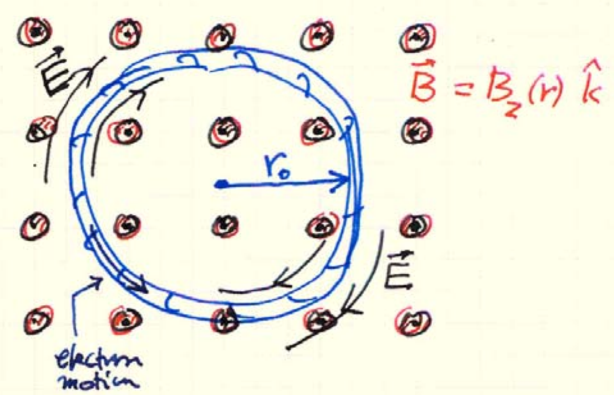
$$E_\phi(r,t) = -\frac{r}{2}\frac{d\overline{B}}{dt}$$

The Betatron

↳ an <u>electron accelerator</u> which uses electromagnetic induction to accelerate electrons to high energy.



Top View



As B_z increases, there is an induced electric field, clockwise.

$$\vec{E} = E_\phi \hat{\phi} \quad \text{where} \quad E_\phi(r,t) = -\frac{r}{2}\frac{d\bar{B}}{dt}$$

The induced electric field accelerates the electrons, which circulate counterclockwise in the vacuum tube.

<u>The electrons' motion</u>

$$\frac{d\vec{p}}{dt} = \vec{F} \qquad \text{where} \quad \vec{p} = p_\phi(t)\,\hat{\phi}(t)$$

$d\hat{\phi} = -\hat{r}\,\omega dt$

$\omega = v/r_0$

$$\frac{d\vec{p}}{dt} = \frac{dp_\phi}{dt}\hat{\phi} + p_\phi\frac{d\hat{\phi}}{dt} = \frac{dp_\phi}{dt}\hat{\phi} - p_\phi\,\omega\,\hat{r}$$

$$\vec{F} = -e\vec{E} - e\vec{v}\times\vec{B} = -eE_\phi\hat{\phi} - evB_z\hat{r}$$

Radial Equation : $p_\phi(t) = e r_0 B_z(r_0, t)$ (1)

Azimuthal Equation : $\frac{dp_\phi}{dt} = -eE_\phi(r_0,t)$ (2)

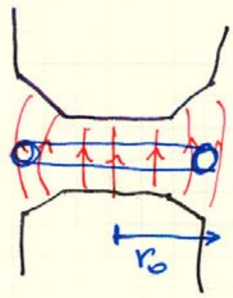
and Faraday's law $E_\phi(r_0,t) = \frac{-r_0}{2}\frac{d\bar{B}}{dt}$

$\Rightarrow \quad \frac{dp_\phi}{dt} = +\frac{e\, r_0}{2}\frac{d\bar{B}}{dt}$

<u>Result</u> The electrons remain at constant radius r_0, provided that

$$B(r_0,t) = \frac{1}{2}\bar{B}(r_0,t)$$

THE BETATRON CONDITION

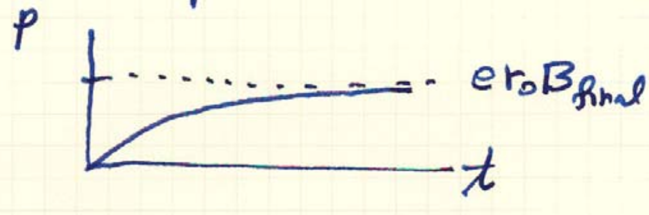


B_z at r_0 is $\frac{1}{2}\times$ the average of B_z inside r_0

$$B(r_0,t) = \frac{1}{2}\frac{\int_0^{r_0} B_z\, dA}{\pi r_0^2} = \frac{1}{2}\frac{\Phi}{\pi r_0^2}$$

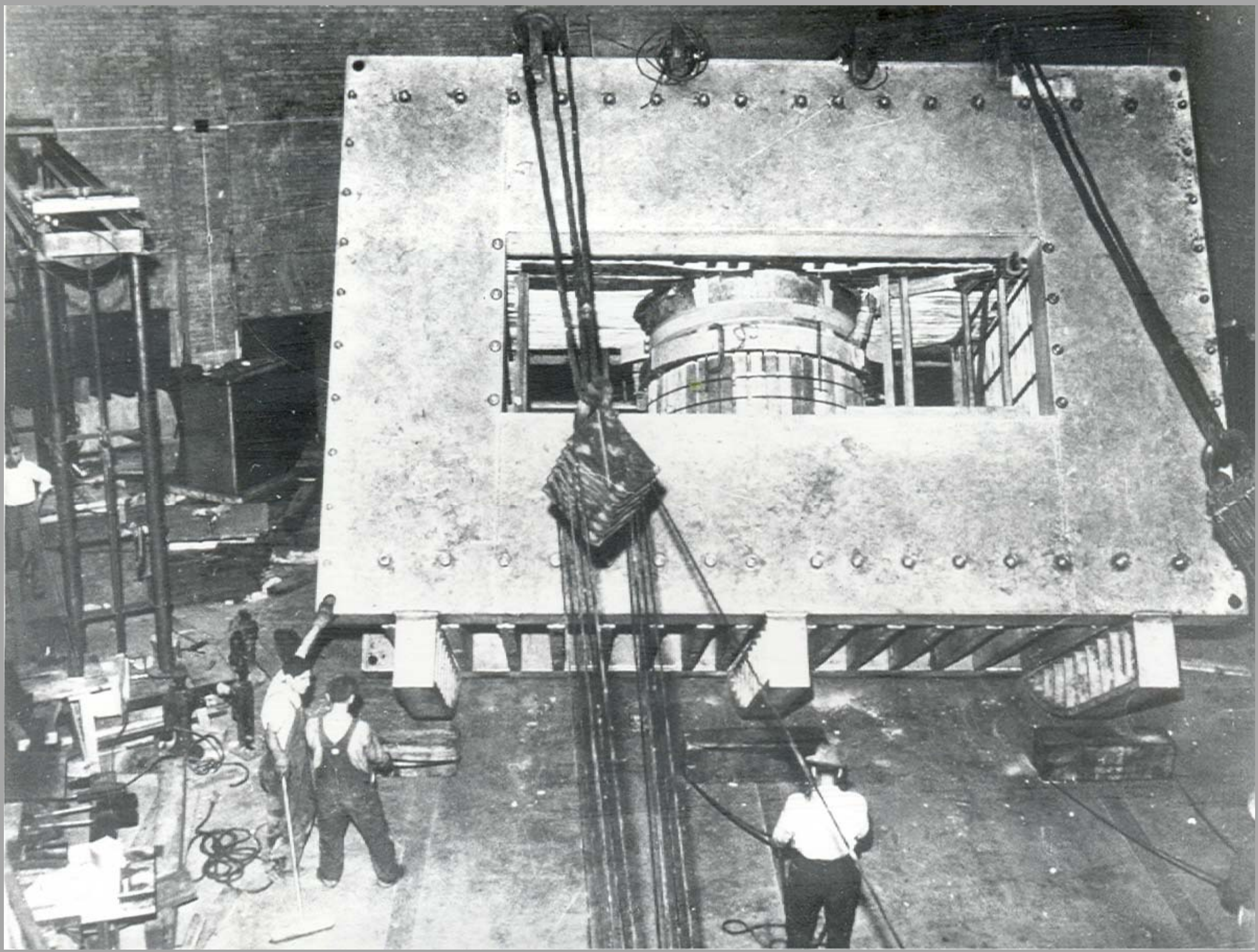
design the pole shapes to satisfy this requirement.

$\longrightarrow$ then $p(t) = e r_0 B(r_0,t)$



First betatron = 6MeV (1942)
Donald Kerst, University of Illinois

In 1950, a 300-MeV betatron, more powerful than that called for in the original design, goes online in its own new building on the corner of Stadium Drive and Oak Street (U of Illinois)



Quiz Question

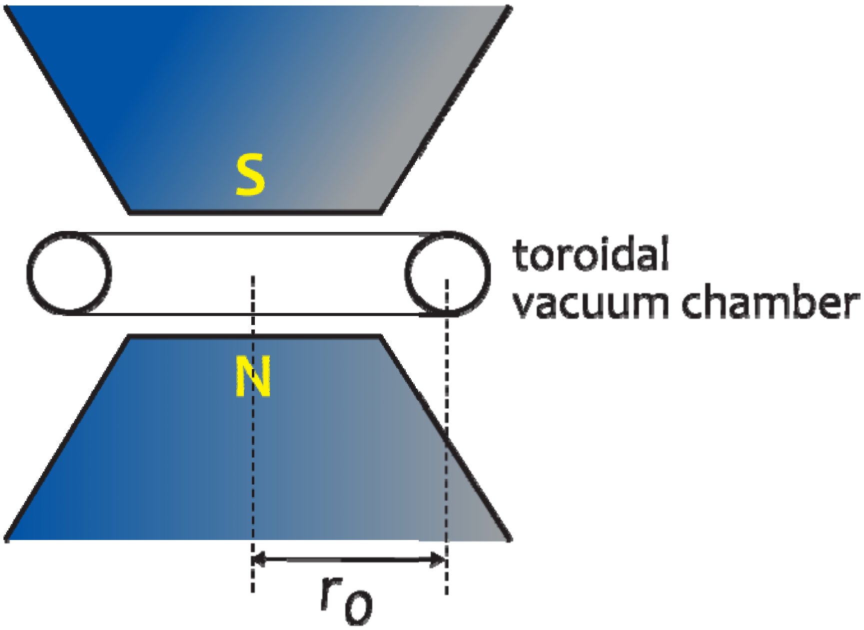
Use these betatron parameters:

- final electron energy = 6 MeV
- final electron momentum = 5.97 MeV/c

  note: $pc = \sqrt{E^2 - (mc^2)^2}$

- radius $r_0 = 20$ cm

Calculate the final magnetic flux $\int B\,dA$ through the area πr_0^2.



Quiz:
Use these betatron parameters:
Electron energy E = 6 MeV
Electron momentum pc = 5.97 MeV
Radius = 10 cm

$$pc = \sqrt{E^2 - (mc^2)^2}$$

Calculate the final magnetic flux through the area $\pi\, r_0^2$.