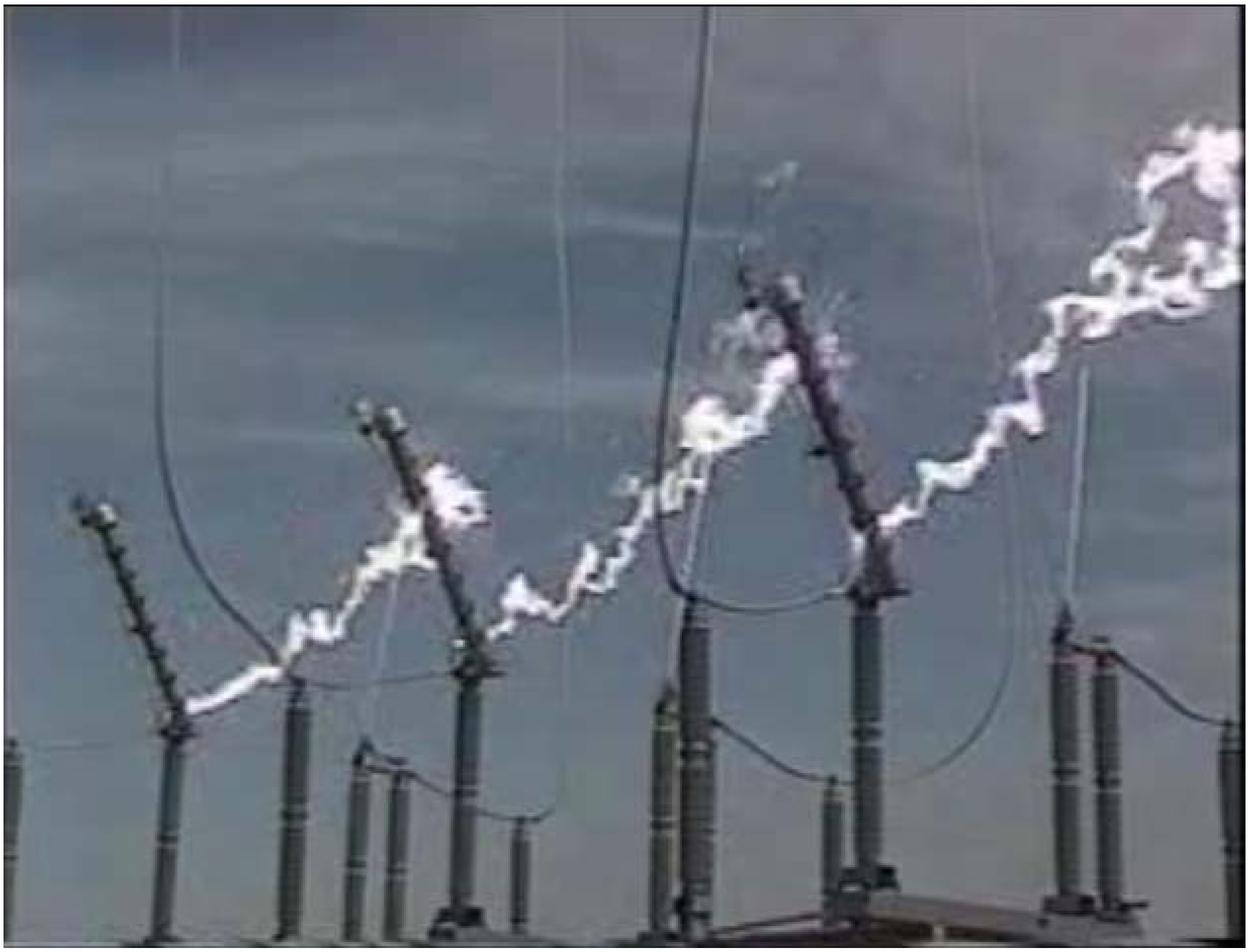
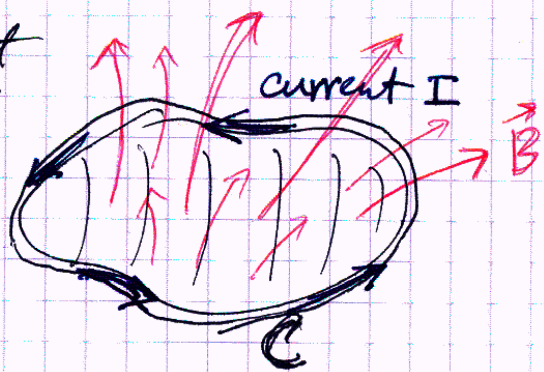
Inductance



- self inductance L
- mutual inductance M

Self inductance

- Abstract



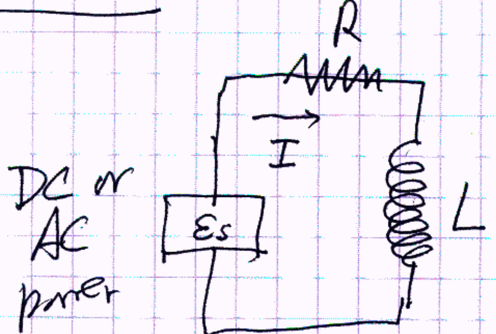
Ampere's Law $\Rightarrow$

$$\Phi = L I$$

Faraday's Law $\Rightarrow$

$$\mathcal{E} = -L\frac{dI}{dt}$$

- Practical



$$\mathcal{E}_s - L\frac{dI}{dt} = IR$$

"back emf"
Lenz's law

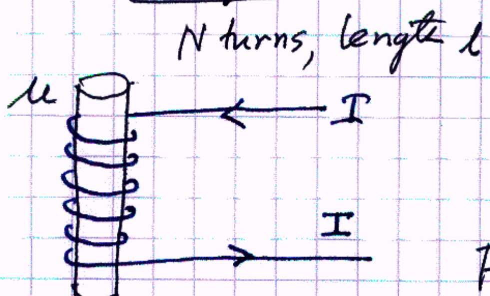
$$L\frac{dI}{dt} + IR = \mathcal{E}_s(t)$$

Suppose $\mathcal{E}_s = 0$. Then

$$\frac{dI}{dt} = -\frac{R}{L}I$$

$$I(t) = I_0 e^{-\frac{R}{L}t}$$

- Example

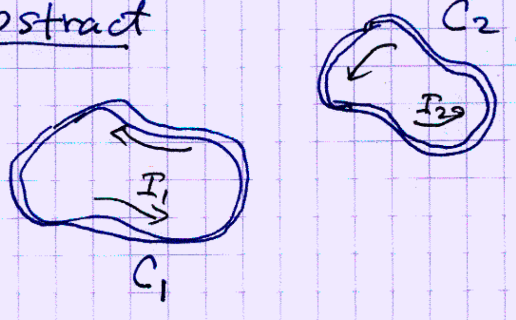


Flux $\Phi = NBA$ where $B = \mu\frac{N}{l}I$

Self inductance $L = \frac{\mu A}{l}N^2$

Mutual inductance

Abstract



$$\Phi_1 = \underbrace{L_1 I_1}_{\substack{\text{flux due to} \\ B_1 \propto I_1}} + \underbrace{M_{12} I_2}_{\substack{\text{flux due to} \\ B_2 \propto I_2}}$$

$$\Phi_2 = \underbrace{L_2 I_2}_{\substack{\text{flux due to} \\ B_2 \propto I_2}} + \underbrace{M_{21} I_1}_{\substack{\text{flux due to} \\ B_1 \propto I_1}}$$

Write $\vec{B}_2 = \nabla \times \vec{A}_2$

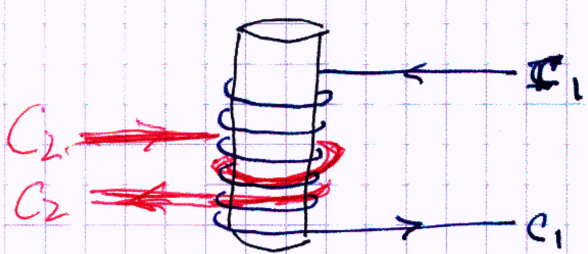
Then $\underbrace{\int_{S_1} \vec{B}_2 \cdot d\vec{A}}_{M_{12} I_2} = \oint_{C_1} \vec{A}_2 \cdot d\vec{l}_1$ by Stokes' theorem

$$\vec{A}_2(\vec{x}) = \frac{\mu_0}{4\pi} \oint_{C_2} \frac{I_2 \, d\vec{l}_2}{|\vec{x} - \vec{x}_2|}$$

$$M_{12} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{x}_1 - \vec{x}_2|} \qquad \text{(Neumann)}$$

$\hookrightarrow$ implies that $M_{12} = M_{21}$

Example



I can easily calculate Φ_{21} (through 2 due to 1)

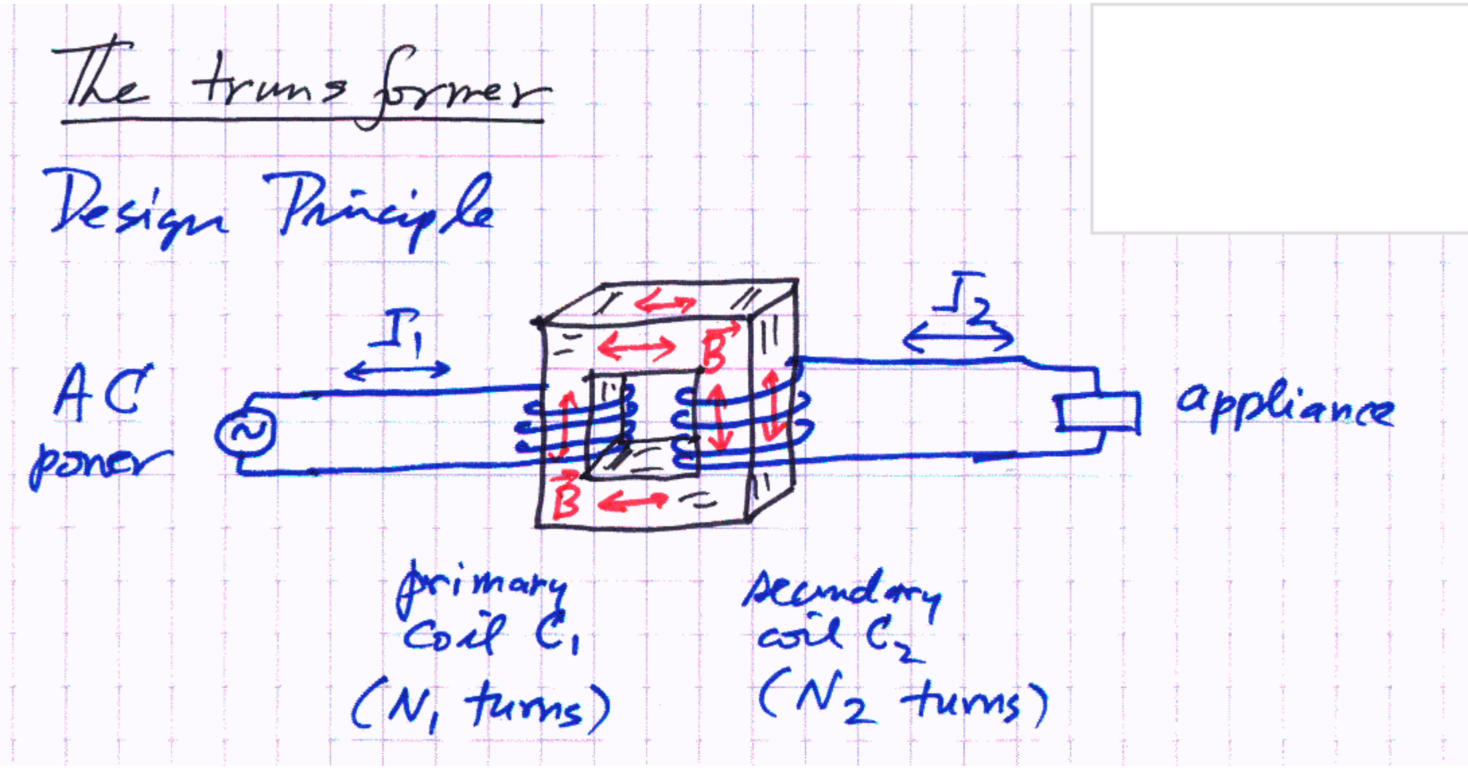
so that tells me M_{21}:

$$M_{21} I_1 = N_2 B_1 A = N_2 \mu_0 \frac{N_1}{l_1} I_1 A$$

$$M_{21} = \frac{\mu_0 N_1 N_2 A}{l_1}$$

I can't easily calculate Φ_{12} but

I know M_{12} = same thing $= \frac{\mu_0 N_1 N_2 A}{l_1}$



Self and Mutual Inductances

$$L_1 = \frac{\Phi}{I} = \frac{N_1 B A}{I} \quad \text{and} \quad B = \mu_0 N_1 I / \ell$$

$$M_{21} = \frac{N_2 B_1 A}{I_1} \quad \text{and} \quad B_1 = \mu_0 N_1 I_1 / \ell$$

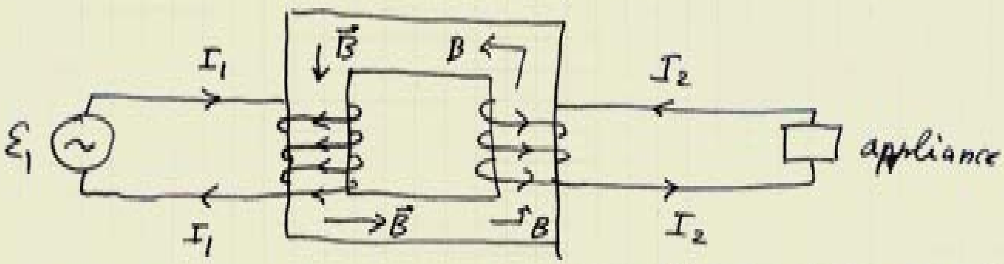
$$L_1 = N_1^2 \lambda \quad ; \quad L_2 = N_2^2 \lambda \quad ; \quad M = N_1 N_2 \lambda$$

$$\boxed{M^2 = L_1 L_2}$$

$$\lambda = \frac{\mu_0 A}{\ell}$$

Transformer Equations

10.5/11



$\mathcal{E}_1 = A_1 \cos\omega t$

ideal: $\begin{cases} L_1 = N_1^2 \lambda, \; L_2 = N_2^2 \lambda \\ M = N_1 N_2 \lambda \; ; \; M^2 = L_1 L_2 \end{cases}$

$$\mathcal{E}_1 - L_1 \frac{dI_1}{dt} - M \frac{dI_2}{dt} = I_1 R_1$$

$$-L_2 \frac{dI_2}{dt} - M \frac{dI_1}{dt} = I_2 R_2$$

the signs must be consistent w/ Lenz's law.

$$A_1 \cos\omega t = L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} + I_1 R_1 \qquad (1)$$

$$0 = L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} + I_2 R_2 \qquad (2)$$

$(1) \times L_2 - (2) \times M \implies$

$$L_2 A_1 \cos\omega t = L_2 R_1 I_1 - M R_2 I_2$$

$$A_1 \cos\omega t = R_1 I_1 - \frac{N_1}{N_2} R_2 I_2$$

$$I_2 = - \frac{N_2}{N_1} \frac{1}{R_2} \left\{ A_1 \cos\omega t - R_I I_1 \right\}$$

$$\frac{dI_2}{dt} = - \frac{N_2}{N_1 R_2} \left\{ -\omega A_1 \sin\omega t - R_1 \frac{dI_1}{dt} \right\}$$

10.5/12

Plug $\frac{dI_2}{dt}$ into equation (1) $\Rightarrow$ ~~an~~ differential equation for $I_1(t)$.

Solution $\quad I_1(t) = a\cos\omega t + b\sin\omega t$

$$R_1 R_2 a + (\omega L_1 R_2 + \omega L_2 R_1)\, b = R_2 A_1 \qquad (3)$$

$$-(\omega L_1 R_2 + \omega L_2 R_1)\, a + R_1 R_2 b = -\omega L_2 A_1 \qquad (4)$$

Let $\quad q = \omega L_1 R_2 + \omega L_2 R_1$

Det Matrix $= R_1^2 R_2^2 + q^2$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{R_1^2 R_2^2 + q^2} \begin{bmatrix} R_1 R_2 & -q \\ q & R_1 R_2 \end{bmatrix} \begin{pmatrix} R_2 A_1 \\ -\omega L_2 A_1 \end{pmatrix}$$

$$a = \frac{(R_1 R_2^2 + q\,\omega L_2)\, A_1}{R_1^2 R_2^2 + q^2}$$

units: V/Ω ✓ $= A$

$$b = \frac{(q R_2 - R_1 R_2 \omega L_2)\, A_1}{R_1^2 R_2^2 + q^2}$$

V/Ω ✓ $= A$

10.5/13

Special ideal case : $R_1 = 0$.

$$a = \frac{\omega L_2}{q} A_1 = \frac{L_2}{L_1 R_2} A_1 = \frac{N_2^2}{N_1^2} \frac{A_1}{R_2}$$

$$b = \frac{R_2}{q} A_1 = \frac{A_1}{\omega L_1}$$

$$I_2 = - \frac{N_2}{N_1} \frac{A_1}{R_2} \cos \omega t$$

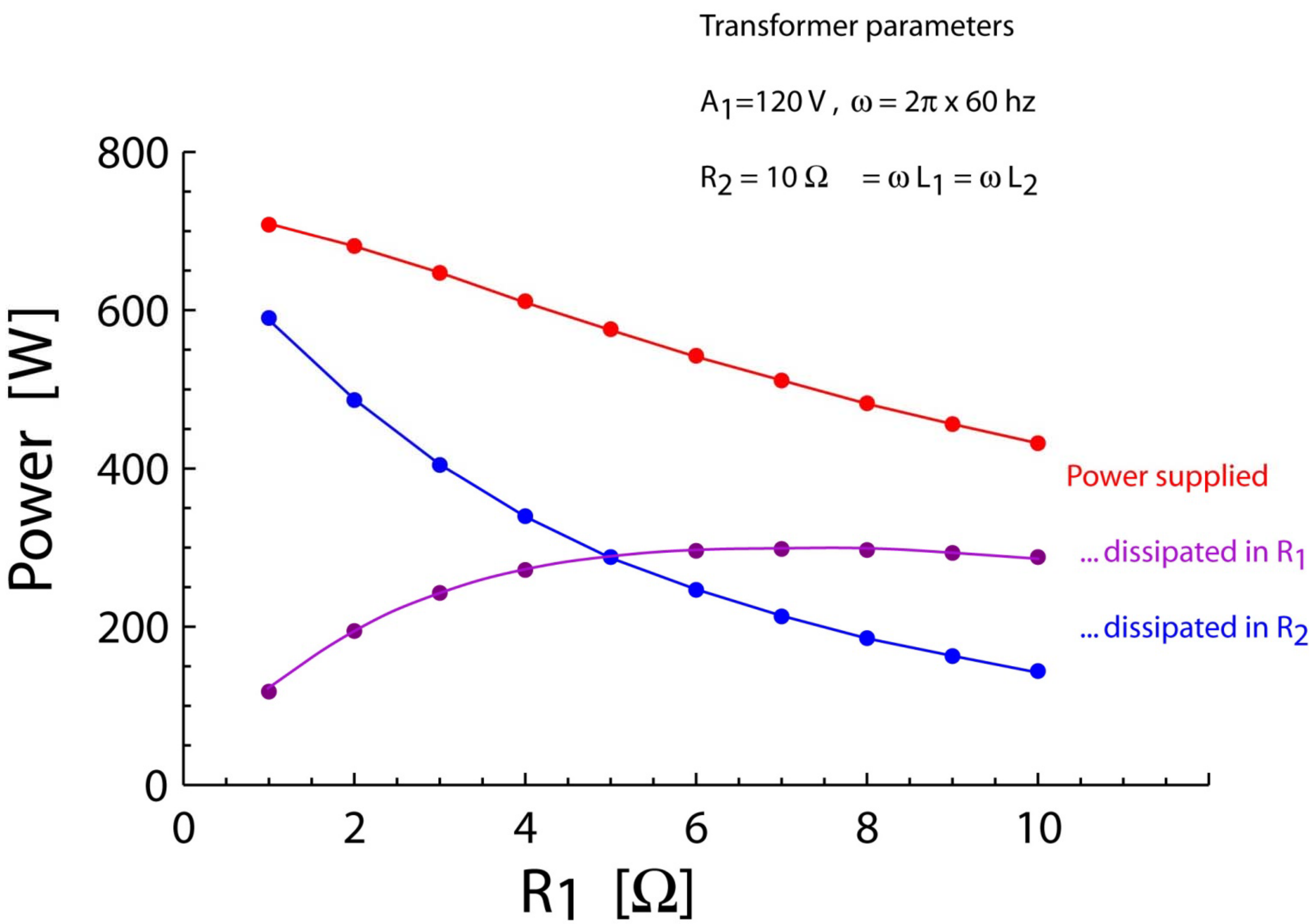
Exercise $\langle \mathcal{E}_1 I_1 \rangle = \langle I_2^2 R_2 \rangle$ energy is conserved

$$\langle \mathcal{E}_1 I_1 \rangle \to \tfrac{1}{2} A_1 a \qquad \langle I_2^2 R_2 \rangle \to \left(\frac{N_2}{N_1}\right)^2 \frac{A_1^2}{R_2} \frac{1}{2} \quad ✓$$

Exercise $I_2 R_2 = - \frac{N_2}{N_1} \mathcal{E}_1$ ✓

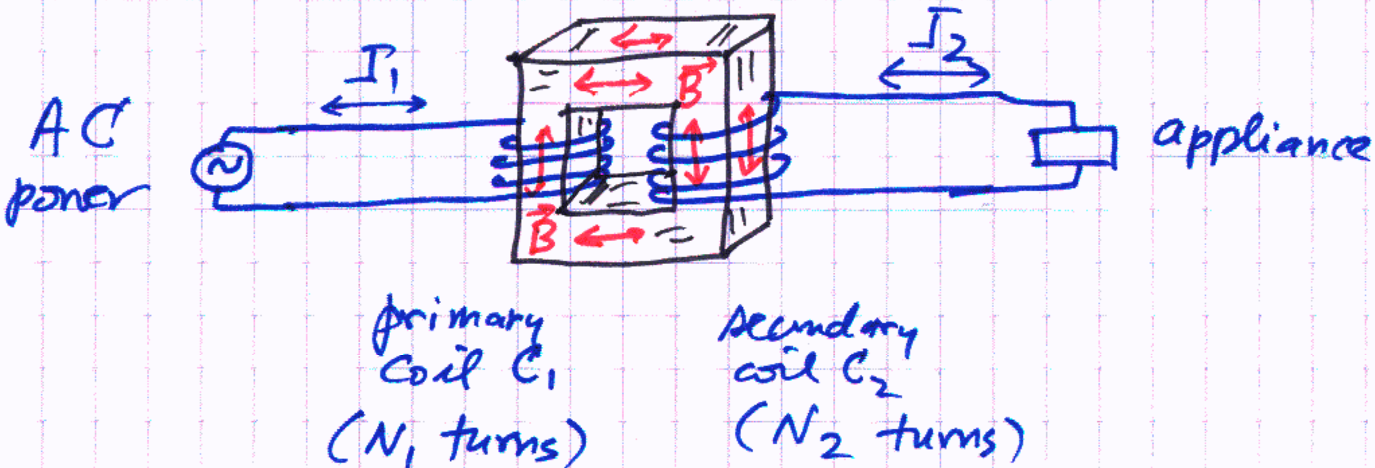
IDEAL TRANSFORMER EQUATIONS

… needs a computer!



The transformer

Design Principle



Quiz

For times $t < 0$, the switch is open, as shown. For times $t \geq 0$ the switch is closed.
(a) Determine the current I(t) around the circuit for $t \geq 0$.
(b) Now suppose $R = 10\ \Omega$ and $L = 0.025$ H. Calculate the time when I is 90% of its final value, $I(t) = 0.9\ \mathcal{E}/R$. Express the answer in milliseconds.