Electric and magnetic units …

$$F = q\,(E + v \times B)$$

$$[E] \;=\; m/s\,[B]$$

$$V/m \;=\; Tm/s$$

Electromagnetic Waves in Vacuum 11.2/1

Maxwell's equations

$$\nabla \cdot \vec{E} = 0 \quad \text{and} \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{and} \quad \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

We'll construct the general polarized plane wave

$$\vec{E}(\vec{x}, t) = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

Complex exponentials

Of course $\vec{E}$ must be a real function. But it is more convenient to solve the equations using complex exponential functions. Just remember, at the end of the calculation we must take the real part (or the imaginary part) to get a real solution

$$e^{i\theta} = \cos\theta + i \sin\theta \quad \text{(L. Euler)}$$

or

$$A e^{i\theta} = |A| \left\{ \cos(\theta + \delta) + i \sin(\theta + \delta) \right\}$$

where $A = |A| \, e^{i\delta}$

So, $\vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ means the Real Part

$$\hookrightarrow \vec{E}_0 \cos(\vec{k}\cdot\vec{x} - \omega t)$$

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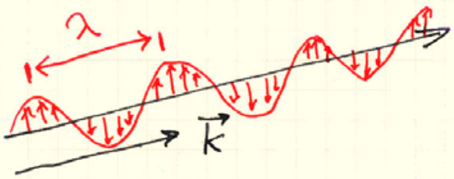
Polarized plane wave

$$\vec{E}(\vec{x},t) = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

- Harmonic time dependence; period $T = \frac{2\pi}{\omega}$
- Propagation in the direction of $\vec{k}$.



$x_{\parallel} = \vec{k}\cdot\vec{x}/k$

- Wavelength $\lambda = \frac{2\pi}{k}$

$$e^{ik(x_{\parallel}+\lambda)} = e^{ikx_{\parallel}} \underbrace{e^{2\pi i}}_{1} = e^{ikx_{\parallel}}$$

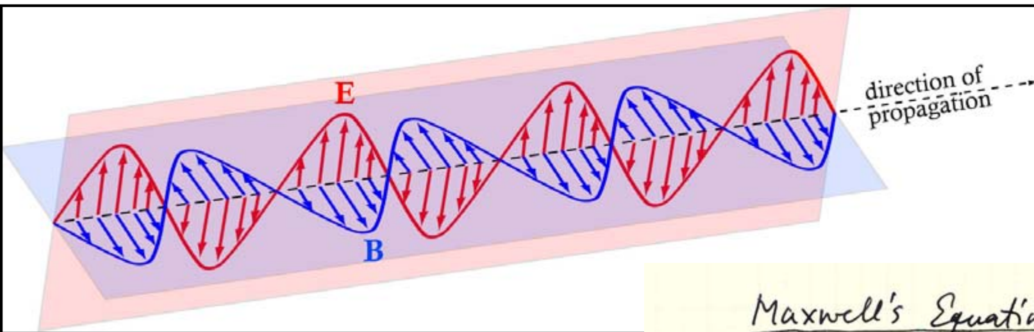
- Phase velocity

$$\vec{k}\cdot\vec{x} - \omega t = \text{constant}$$

$$k x_{\parallel} - \omega t = \text{constant}$$

$$k \, \Delta x_{\parallel} - \omega \, \Delta t = 0$$

$$v_{phase} = \frac{\Delta x_{\parallel}}{\Delta t} = \frac{\omega}{k}$$



Maxwell's Equations must be satisfied

$$\vec{E}(\vec{x},t) = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

(1) $\nabla\cdot\vec{E} = 0$

$$\nabla\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$
$$= (E_{0x}\, ik_x + E_{0y}\, ik_y + E_{0z}\, ik_z)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$= i\, \vec{k}\cdot\vec{E}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$= 0$$

$$\boxed{\vec{k}\cdot\vec{E}_0 = 0}$$

The electric field oscillates in a direction perpendicular to the direction of propagation.

Electromagnetic waves are TRANSVERSE waves.

(2) $\nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$

$$\nabla\times\vec{E} = (i\vec{k}\times\vec{E}_0)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

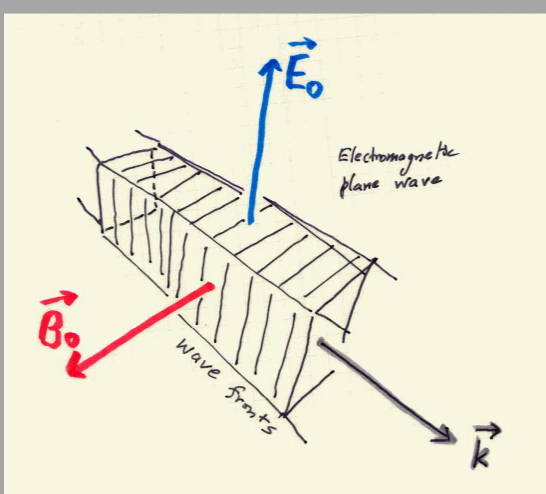
$$\frac{\partial}{\partial x_i} = ik_i$$

$$\frac{\partial}{\partial t} = -i\omega$$

Therefore $\vec{B} = \dfrac{\vec{k}\times\vec{E}_0}{\omega}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$$\vec{B}(\vec{x},t) = \vec{B}_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

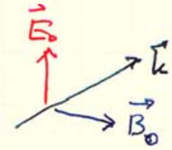
$$\boxed{\vec{B}_0 = \frac{\vec{k}\times\vec{E}_0}{\omega}}$$



Note $\vec{E}_0 = \dfrac{\omega}{k^2}\vec{B}_0\times\vec{k}$

Also $\vec{E}_0\times\vec{B}_0 = \dfrac{\vec{k}}{\omega}E_0^2$

$\vec{k}$, $\vec{E}_0$, and $\vec{B}_0$ form an orthogonal triad.



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(3) $\nabla \cdot \vec{B} = 0$

This equation is already satisfied:

$\vec{B} = \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$\nabla \cdot \vec{B} = i\vec{k} \cdot \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ $\qquad \boxed{\dfrac{\partial}{\partial x_i} = i k_i}$

But $\vec{k} \cdot \vec{B}_0 = 0$ because $\vec{B}_0 = \underbrace{\dfrac{\vec{k} \times \vec{E}_0}{\omega}}_{\text{is perpendicular to } \vec{k}}$

(4) $\nabla \times \vec{B} = \mu_0 \epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

$$i\vec{k} \times \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)} = \mu_0 \epsilon_0 (-i\omega) \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\vec{k} \times \vec{B}_0 = -\mu_0 \epsilon_0 \omega \vec{E}_0$$

Now, $\vec{B}_0 = \dfrac{\vec{k} \times \vec{E}_0}{\omega}$ so $\vec{k} \times \vec{B}_0 = \dfrac{\vec{k} \times (\vec{k} \times \vec{E}_0)}{\omega}$

$$= \frac{1}{\omega} \left\{ \vec{k} (\vec{k} \cdot \vec{E}_0) - k^2 \vec{E}_0 \right\}$$

$$= -\frac{k^2}{\omega} \vec{E}_0$$

So $\quad -\dfrac{k^2}{\omega} \vec{E}_0 = -\mu_0 \epsilon_0 \omega \vec{E}_0$

$$\boxed{\frac{\omega^2}{k^2} = \frac{1}{\mu_0 \epsilon_0}}$$

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<u>Electromagnetic waves in vacuum</u>

$$v_{phase} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$$

$$v_{group} = \frac{d\omega}{dk} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3.00 \times 10^8 \text{ m/s}$$

- All frequencies have the same speed.
- The wave speed does not depend on the frame of reference ← assuming Maxwell's equations have the same form in all inertial frames. (Einstein's theory of relativity)
- $B_0 = \frac{E_0}{c}$, independent of frequency

The solution constructed here is called the polarized plane wave. It is an <u>idealized</u> e.m. wave: perfectly polarized and coherent. The wave fronts are infinite planes $\perp$ to $\vec{k}$. (coherence)

The electric field oscillates only in one direction. (polarized)

(The magnetic field oscillates in a $\perp$ direction.)

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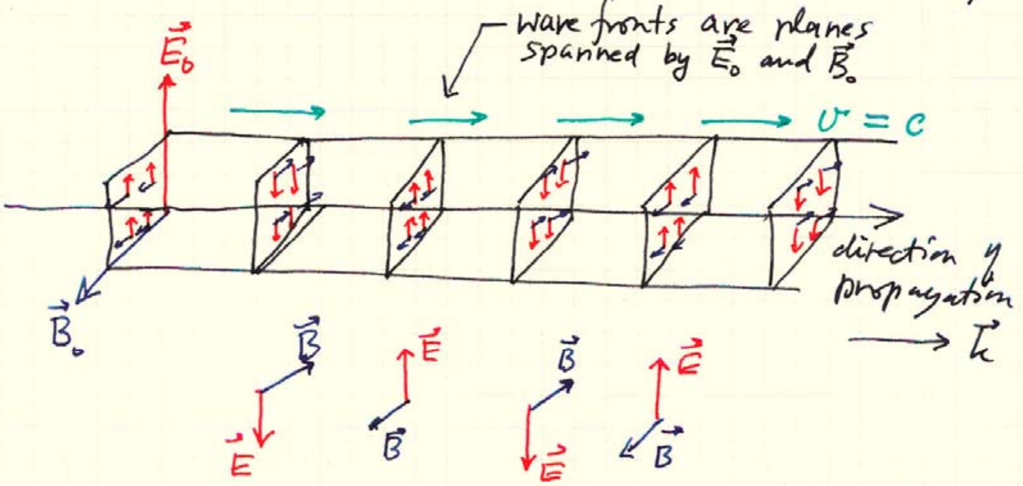


$\vec{E} \times \vec{B}$ is everywhere in the direction of $\vec{k}$.

On any plane $\perp$ to $\vec{k}$, $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ are <u>independent of $\vec{x}$</u> (oscillating in t)

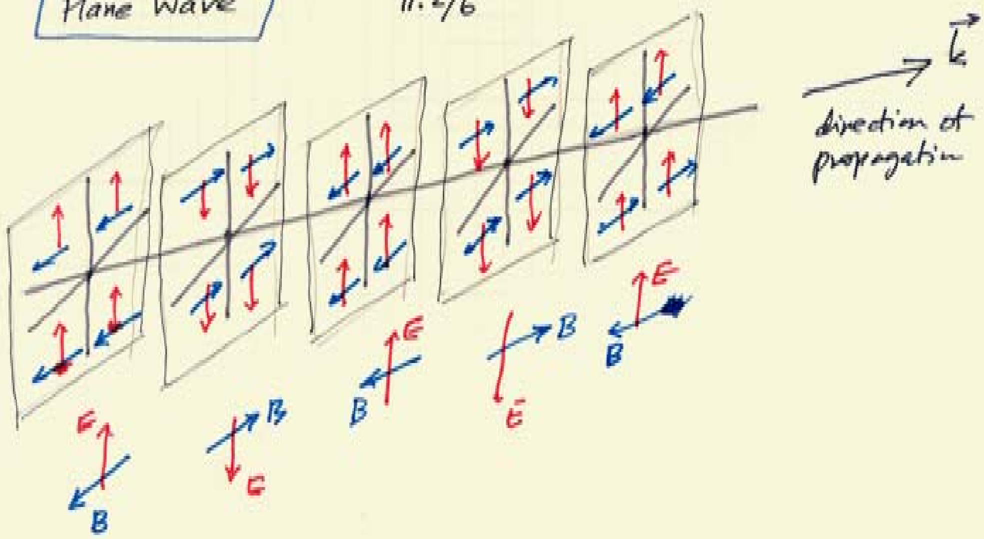
↳ coherence

The ideal polarized plane waves are important because they are a <u>complete set</u> of solutions to Maxwell's equations.

<u>Any</u> solution is a superposition of plane waves.

<u>Example</u> A finite pulse of light (a <u>flash</u>)

Plane Wave   11.2/6



• $\vec{E}\times\vec{B}$ is everywhere in direction of $\vec{k}$. (flow of energy)

• On any plane ⊥ to $\vec{k}$, $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ are independent of $\vec{x}$ (oscillating in t) → "coherence"

The ideal polarized plane waves are important because they are a complete set of solutions to Maxwell's equations. → "completeness"

Any solution can be written as a superposition of plane waves. → "superposition principle"

Example A finite pulse of light (a flash)

Quiz Question

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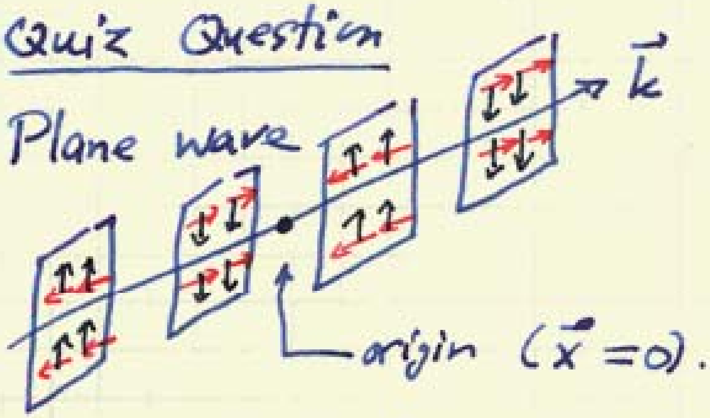
Plane wave



Energy densities in an electromagnetic plane wave .....

(A) Determine $u_E(\vec{x},t)$ at $\vec{x}=0$.

(B) Determine $u_M(\vec{x},t)$ at $\vec{x}=0$.

Express both answers in terms of E_0;

$E_0 = |\vec{E}_0|$.