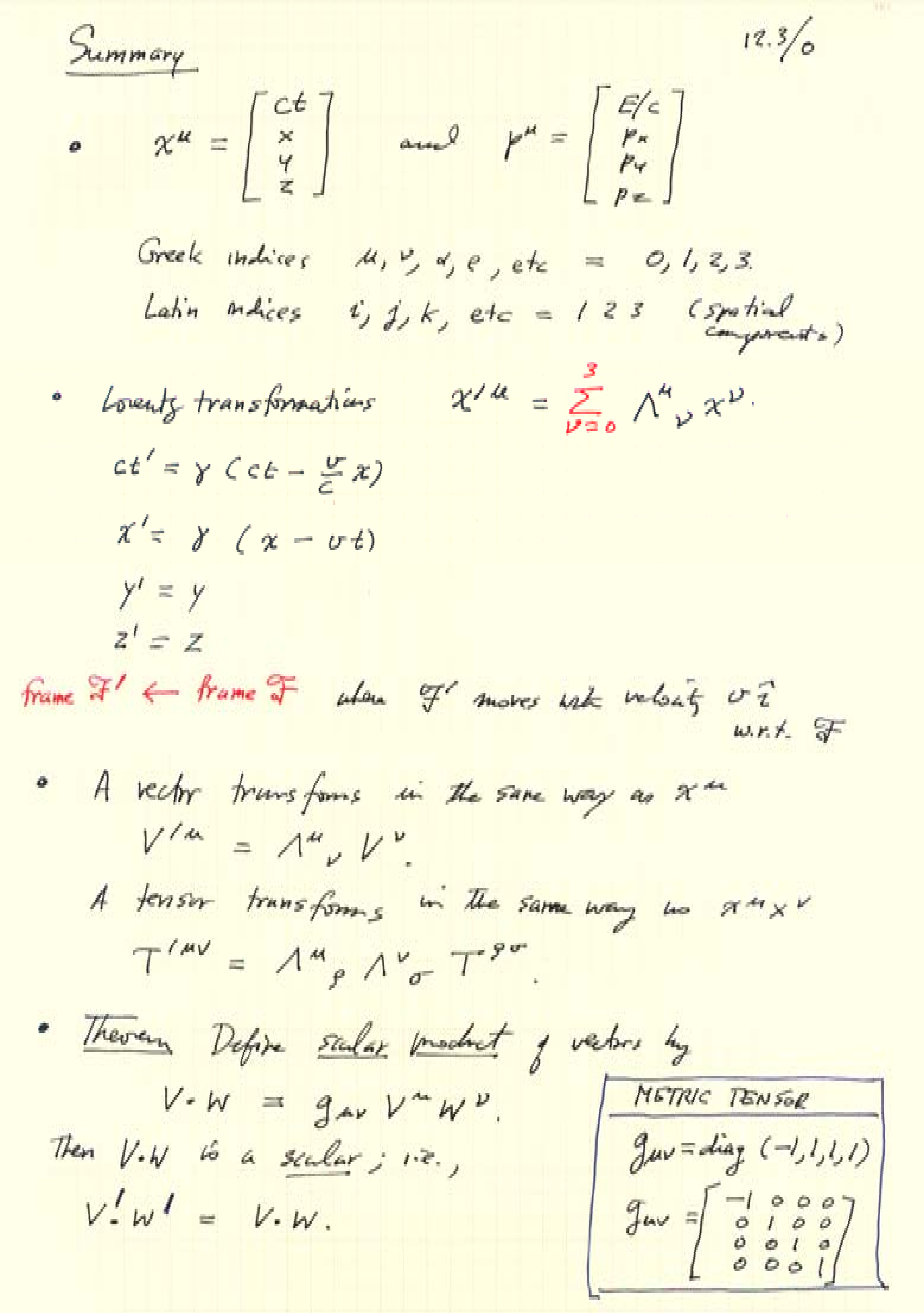


Summary

12.3/0

- $x^\mu = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$ and $p^\mu = \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix}$

  Greek indices μ, ν, α, ρ, etc $= 0, 1, 2, 3$.

  Latin indices i, j, k, etc $= 1\ 2\ 3$ (spatial components)

- Lorentz transformations $x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu$.

$$ct' = \gamma\left(ct - \frac{v}{c}x\right)$$

$$x' = \gamma\,(x - vt)$$

$$y' = y$$

$$z' = z$$

frame $\mathcal{F}'$ ← frame $\mathcal{F}$ where $\mathcal{F}'$ moves with velocity $v\hat{i}$ w.r.t. $\mathcal{F}$

- A vector transforms in the same way as x^μ

$$V'^\mu = \Lambda^\mu{}_\nu V^\nu.$$

  A tensor transforms in the same way as $x^\mu x^\nu$

$$T'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma}.$$

- Theorem Define scalar product of vectors by

$$V \cdot W = g_{\mu\nu} V^\mu W^\nu.$$

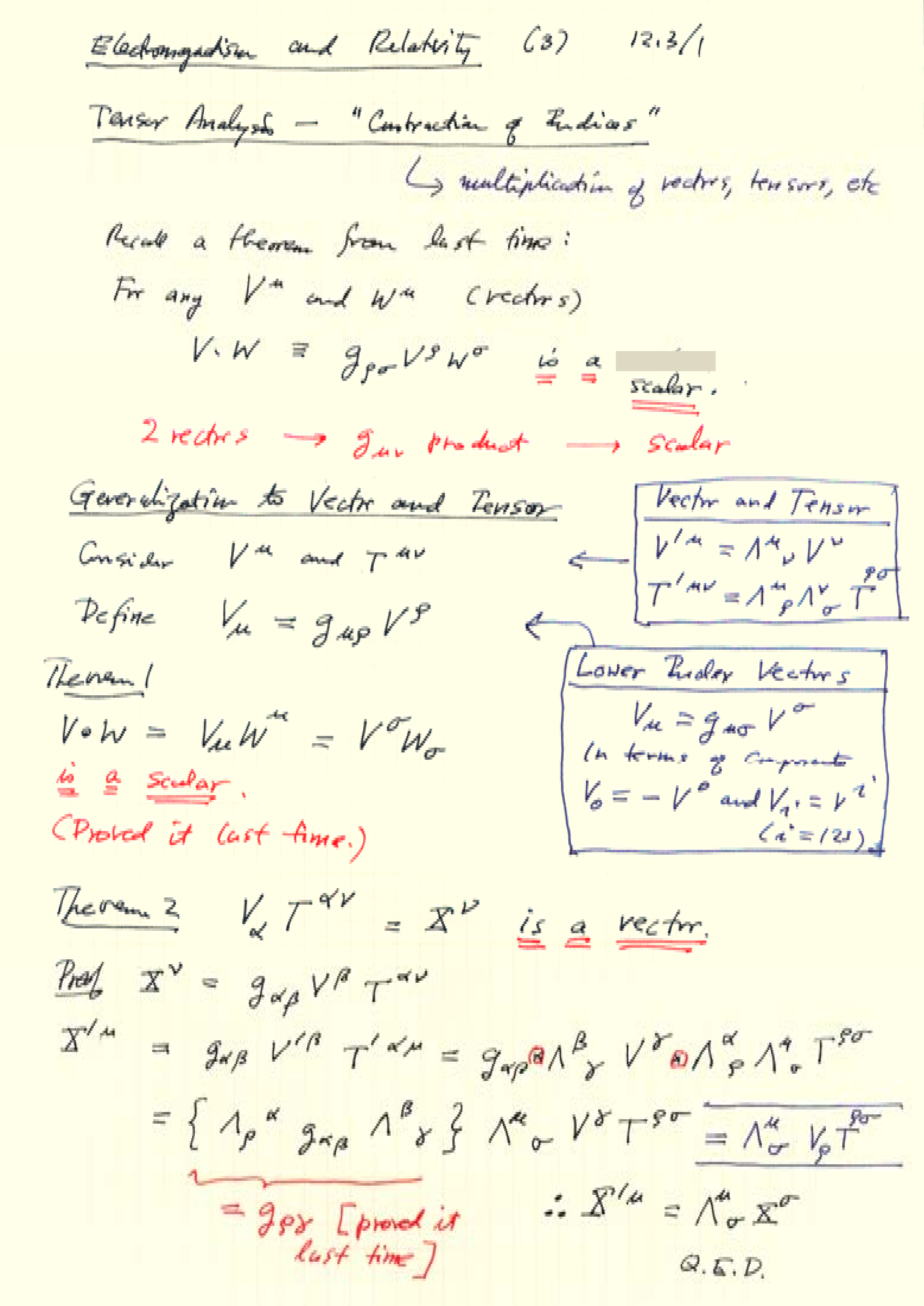
Then $V \cdot W$ is a scalar; i.e.,

$$V' \cdot W' = V \cdot W.$$

METRIC TENSOR

$$g_{\mu\nu} = \text{diag}\,(-1, 1, 1, 1)$$

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Electromagnetism and Relativity (3) 12.3/1

Tensor Analysis — "Contraction of Indices"

↳ multiplication of vectors, tensors, etc

Recall a theorem from last time:

For any V^μ and W^μ (vectors)

$$V \cdot W \equiv g_{\rho\sigma} V^\rho W^\sigma$$ is a [redacted] scalar.

2 vectors → $g_{\mu\nu}$ product → scalar

Generalization to Vector and Tensor

Consider V^μ and $T^{\mu\nu}$

Define $V_\mu = g_{\mu\rho} V^\rho$

Vector and Tensor

$$V'^\mu = \Lambda^\mu{}_\nu V^\nu$$

$$T'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma}$$

Lower Index Vectors

$$V_\mu = g_{\mu\sigma} V^\sigma$$

(In terms of Components

$V_0 = -V^0$ and $V_i = V^i$

$(i = 123)$

Theorem 1

$$V \cdot W = V_\mu W^\mu = V^\sigma W_\sigma$$

is a scalar.

(Proved it last time.)

Theorem 2 $V_\alpha T^{\alpha\nu} = X^\nu$ is a vector.

Proof $X^\nu = g_{\alpha\beta} V^\beta T^{\alpha\nu}$

$$X'^\mu = g_{\alpha\beta} V'^\beta T'^{\alpha\mu} = g_{\alpha\beta} \Lambda^\beta{}_\gamma V^\gamma \Lambda^\alpha{}_\rho \Lambda^\mu{}_\sigma T^{\rho\sigma}$$

$$= \underbrace{\{ \Lambda_\rho{}^\alpha g_{\alpha\beta} \Lambda^\beta{}_\gamma \}}_{= g_{\rho\gamma} \text{ [proved it last time]}} \Lambda^\mu{}_\sigma V^\gamma T^{\rho\sigma} = \Lambda^\mu{}_\sigma V_\rho T^{\rho\sigma}$$

$$\therefore X'^\mu = \Lambda^\mu{}_\sigma X^\sigma$$

Q.E.D.

Construct covariant equations

12.3/2

Recall another theorem from last time:
An equation is "manifestly covariant" if it is written in tensor form:

e.g., $T_1^{\mu\nu} = T_2^{\mu\nu}$ implies $T_1'^{\mu\nu} = T_2'^{\mu\nu}$

(in some frame $\mathcal{F}$) (in any other frame $\mathcal{F}'$)

Therefore our goal is to write the equations of electromagnetism in tensor form. (COVARIANT)

The Electromagnetic Field

How are $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ defined?

- $\vec{E}(\vec{x},t) = \dfrac{\vec{F}}{q}$ for a test charge q at rest at $\vec{x}$.

  i.e., $\vec{F}_{\text{electric}} = q\,\vec{E}(\vec{x},t)$.

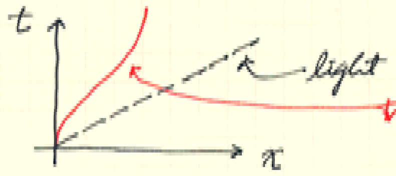
- Similarly, $\vec{F}_{\text{magnetic}} = q\,\vec{u} \times \vec{B}$ where $\vec{u}$ = the particle velocity

$\Rightarrow$ Newton's second law,

$$m\frac{d\vec{p}}{dt} = \vec{F} = q\left\{ \vec{E}(\vec{x}) + \vec{u} \times \vec{B}(\vec{x}) \right\}.$$

BUT, that's not "manifestly covariant"!

Particle dynamics in special relativity 12.3/8



light cone (x = ct)

trajectory of a particle = a curve in space time

Let $\xi^\mu(\tau)$ = the trajectory of the particle (q) as a function of the proper time of q

Recall the definition of proper time ...

$$c^2(d\tau)^2 = (d\xi^0)^2 - (d\xi^1)^2 - (d\xi^2)^2 - (d\xi^3)^2$$

or $$c^2(d\tau)^2 = -g_{\mu\nu}\, d\xi^\mu\, d\xi^\nu = -d\xi\cdot d\xi$$

or $$c^2(d\tau)^2 = c^2(dt)^2 - u^2(dt)^2 = \left(1-\frac{u^2}{c^2}\right)(d\xi^0)^2$$

$\boxed{\text{Velocity in frame } \mathcal{F}: \ \vec{u} = \frac{d\vec{\xi}}{dt}}$

$\boxed{\xi^0 = ct}$

4-velocity

$$\eta^\mu = \frac{d\xi^\mu}{d\tau}$$ ← definition; a Lorentz vector

Note that $$\eta^0 = \frac{d\xi^0}{d\tau} = \frac{c\,dt}{\sqrt{c^2-u^2}\,\frac{dt}{c}} = \frac{c}{\sqrt{1-u^2/c^2}} = \gamma c$$

$\boxed{\eta^0 = \gamma c}$

and $$\eta^i = \frac{d\xi^i}{d\tau} = \frac{u^i\,dt}{\sqrt{c^2-u^2}\,\frac{dt}{c}} = \frac{u^i}{\sqrt{1-u^2/c^2}} = \gamma u^i$$

$\boxed{\eta^i = \gamma u^i}$

4-momentum $p^\mu = m\eta^\mu$ ← definition; a Lorentz vector

Recall $$p^0 = \frac{E}{c} = \frac{mc}{\sqrt{1-u^2/c^2}} = \gamma mc$$

$$p^i = m\eta^i = \gamma m u^i$$

$$\boxed{\gamma = \frac{1}{\sqrt{1-u^2/c^2}}}$$

12.3/4

Now, the <u>equation of motion</u> (in covariant form) is

$$\frac{dp^\mu}{d\tau} = K^\mu \quad , \text{ the Minkowski force}$$

(dp^μ = vector ; $d\tau$ = scalar ; $\therefore K^\mu$ must be a Lorentz vector)

So, what is K^μ for the electromagnetic force?
We know that $\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$. So K^μ should also be <u>linear</u> in the fields, and involve the velocity.
That suggests:

$\boxed{\text{Theorem}}$ $K^\mu = q F^{\mu\nu} \eta_\nu$ (*)
where $F^{\mu\nu}$ is the electromagnetic field tensor

(If $F^{\mu\nu}$ is a tensor then K^μ is a vector by earlier Theorem.)

$\boxed{\text{Corollary}}$ $F^{\mu\nu}$ must transform as a tensor.

<u>Construction of $F^{\mu\nu}$</u>

12,3/5

Construction of $F^{\mu\nu}$

Start with the definitions of $\vec{E}$ and $\vec{B}$

$$\frac{dp^i}{dt} = F^i = qE^i + q\underbrace{\epsilon_{ijk}}_{\sum_{j,k=1}^{3} \text{ is implied !}} u^j B^k$$

Now $$\frac{dp^i}{dt} = \frac{dp^i}{d\tau}\frac{d\tau}{dt} = K^i \boxed{\times} \frac{1}{\gamma}$$

Remember?
$$d\tau = \left(1 - \frac{u^2}{c^2}\right)^{1/2} dt = \frac{dt}{\gamma}$$

So $$K^i = \gamma q \left\{ E^i + \epsilon_{ijk} u^j B^k \right\}$$

$$= q\left\{ \frac{\eta_0}{c} E^i + \epsilon_{ijk} \eta^j B^k \right\} \quad ①$$

Remember?
$$\eta^0 = \gamma c$$
$$\eta^i = \gamma u^i$$

We require [by (*) with $\mu = i$]

$$K^i = q F^{i\nu} \eta_\nu$$

$$K^i = q\left[-\eta^0 F^{i0} + \eta^j F^{ij} \right] \quad ②$$

Comparing ① and ②, evidently

$$F^{i0} = -\frac{E^i}{c} \quad \text{and} \quad F^{ij} = \epsilon_{ijk} B^k$$

Result so far 12.3/6

$$F^{\mu\nu} = \begin{array}{c|cccc} \mu \backslash \nu & 0 & 1 & 2 & 3 \\ \hline 0 & ? & ? & ? & ? \\ 1 & -\frac{E_x}{c} & 0 & B_z & -B_y \\ 2 & -\frac{E_y}{c} & -B_z & 0 & B_x \\ 3 & -\frac{E_z}{c} & B_y & -B_x & 0 \end{array}$$

$\underbrace{}_{F^{i0}}$ $\underbrace{}_{F^{ij}}$

$i = 1\,2\,3$
$j = 1\,2\,3$

Lemma The tensor $F^{\mu\nu}$ must be antisymmetric; that is, $F^{\nu\mu} = -F^{\mu\nu}$.

Therefore...

$$F_{\mu\nu} = \begin{array}{c|cccc} \mu \downarrow \; \nu \rightarrow & & & & \\ & 0 & +E_x/c & +E_y/c & +E_z/c \\ & -\frac{E_x}{c} & 0 & B_z & -B_y \\ & -\frac{E_y}{c} & -B_z & 0 & B_x \\ & -\frac{E_z}{c} & B_y & -B_x & 0 \end{array} \qquad (\star)$$

Proof of the Lemma $\boxed{p^\mu = m\eta^\mu}$

$\eta^0 = \gamma c$
$\eta^i = \gamma u^i$

$$p^\mu p_\mu = m\eta^\mu m\eta_\mu = m^2 \eta\cdot\eta = -m^2c^2$$

$$\eta^\mu \eta_\mu = \eta\cdot\eta = -(\eta^0)^2 + (\vec{\eta})^2 = -\gamma^2c^2 + \gamma^2u^2 = -c^2$$

$p^\mu p_\mu$ is a constant, so $\frac{d}{d\tau}(p^\mu p_\mu) = 0$

$$\hookrightarrow \frac{dp^\mu}{d\tau} p_\mu + p^\mu \frac{dp_\mu}{d\tau} \;\; = 2\frac{dp^\mu}{d\tau} p_\mu$$ [redacted]

Thus $K^\mu p_\mu = 0$

And $K^\mu = qF^{\mu\nu}\eta_\nu$, so we require $F^{\mu\nu}\eta_\mu\eta_\nu = 0$

Q.E.D. the lemma. [[An antisymmetric tensor $(F^{\mu\nu})$ contracted with a symmetric tensor $(\eta_\mu\eta_\nu)$ is 0.]]

$$F^{\mu\nu}\eta_\mu\eta_\nu = F^{\mu\nu}\eta_\nu\eta_\mu = F^{\sigma\rho}\eta_\rho\eta_\sigma = -F^{\rho\sigma}\eta_\rho\eta_\sigma = -F^{\mu\nu}\eta_\mu\eta_\nu \quad \therefore F^{\mu\nu}\eta_\mu\eta_\nu = 0$$

12.3/7

Electromagnetic field tensor

$$F^{\mu\nu} = \begin{array}{l} \mu \quad \nu \longrightarrow \\ \downarrow \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \end{array}$$

$$F^{0i} = \frac{E^i}{c} \quad \text{and} \quad F^{i0} = -\frac{E^i}{c} \qquad (i = 1\,2\,3)$$

$$F^{ij} = \epsilon_{ijk} B^k \qquad \Big(\sum_{k=1}^{3} \text{implied} \; ; \quad ij = 1\,2\,3 \Big)$$

$$F^{\mu\nu} = 0 \quad \text{if} \quad \mu = \nu$$

$$F^{\nu\mu} = -F^{\mu\nu}$$

Three important results

(1) The Minkowski force for electrodynamics is

$$\frac{dp^\mu}{d\tau} = K^\mu = q\, F^{\mu\nu} \eta_\nu$$

manifestly covariant

(2) The electromagnetic field tensor is

$F^{\mu\nu}(\vec{E}, \vec{B})$ given above.

(3) $F^{\mu\nu}$ must be a tensor, so

$$F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta} \qquad \Big(\sum_{\alpha,\beta=0}^{3} \text{implied} \Big)$$

That will tell us how the fields $\vec{E}$ and $\vec{B}$ transform under Lorentz transformations.

Quiz Question

12.3/8

(A) For a particle with charge q, determine K^0 in terms of $\vec{E}$, $\vec{B}$, and $\vec{u}$.

{ K^0 : the time component of the Minkowski force }

(B) Show that $\frac{cK^0}{\gamma}$ = the work per time done by the electric field on the particle.