Electricity and Magnetism (4) 12.4/1

Particle Dynamics

↳ motion of a particle with mass m and charge q

- $\eta^\mu = \dfrac{d\xi^\mu}{d\tau}$ and $p^\mu = m\eta^\mu$
- $\dfrac{dp^\mu}{d\tau} = q F^{\mu\nu}\eta_\nu$ $\left(\sum_{\nu=0}^{3} \text{ implied}\right)$
- $$F^{\mu\nu} = \begin{array}{cc} & \nu \longrightarrow \\ \mu\downarrow & \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \end{array}$$ the electromagnetic field tensor

- The quantity $F^{\mu\nu}$ must be a tensor; i.e., it must transform as

$$F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$$

frame $\mathcal{F}'$ $\longleftarrow$ frame $\mathcal{F}$

Then the equation of motion, $\dfrac{dp^\mu}{d\tau} = q F^{\mu\nu}\eta_\nu$ is covariant; i.e., consistent with the postulate of relativity.

TODAY: Write Maxwell's Equations in tensor form, using $F^{\mu\nu}$.

Maxwell's Equations

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↳ linear in $F^{\mu\nu}$;

and first order in derivatives.

$$\boxed{\frac{\partial F^{\mu\nu}}{\partial x^\nu}}$$

First, an important theorem of tensor analysis:

$\frac{\partial}{\partial x^\mu}$ is like a lower index vector; i.e.,

$$\underbrace{\frac{\partial}{\partial x^\nu} T^{\alpha\beta\ldots\mu\nu}}_{\text{tensor of rank } N;\ N \text{ indices } \alpha\beta\ldots\mu\nu} = \underbrace{U^{\alpha\beta\ldots\mu}}_{\text{tensor of rank } N-1;\ N-1 \text{ indices } \alpha\beta\ldots\mu} \text{ is a tensor}$$

("contraction" of index ν)

Examples

- ϕ a scalar $\Rightarrow$ $\frac{\partial\phi}{\partial x^\nu}$ a tensor of rank -1 $= A_\nu$ (a lower index vector)
- V^μ a vector $\Rightarrow$ $\frac{\partial}{\partial x^\mu} V^\mu$ a tensor of rank 0 (a scalar)
- $T^{\mu\nu}$ a tensor $\Rightarrow$ $\frac{\partial T^{\mu\nu}}{\partial x^\nu}$ a tensor of rank $1 = U^\mu$ (an upper index vector)
- etc.

So, $\frac{\partial F^{\mu\nu}}{\partial x^\nu}$ is a spacetime vector.

What vector is it?

$$\begin{array}{cl} & \nu \longrightarrow \\ \mu \downarrow & F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix} \end{array} \qquad 12.4/3$$

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu}$$

$\boxed{\text{Case } \mu = 0}$ $$\frac{\partial F^{0\nu}}{\partial x^\nu} = \frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{0i}}{\partial x^i} \qquad \left(\sum_{i=1}^{3} \text{ implied}\right)$$

$$= 0 + \frac{\partial}{\partial x^i}\left(\frac{E^i}{c}\right) = \frac{1}{c}\nabla\cdot\vec{E}$$

$$= \frac{\rho}{c\epsilon_0} \quad \text{by Gauss's law} \quad \nabla\cdot\vec{E} = \rho/\epsilon_0$$

Write $\frac{\partial F^{0\nu}}{\partial x^\nu} = \mu_0 J^0$ where $J^0 = \frac{\rho}{c\,\epsilon_0\mu_0} = c\rho$.

$\boxed{\text{Case } \mu = i \ (=1\,2\,3)}$ $$\frac{\partial F^{i\nu}}{\partial x^\nu} = \frac{\partial F^{i0}}{\partial x^0} + \frac{\partial F^{ij}}{\partial x^j} \qquad \left(\sum_j\right)$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{-E^i}{c}\right) + \epsilon_{ijk}\frac{\partial}{\partial x_j}B^k$$

$$= \frac{-1}{c^2}\frac{\partial E^i}{\partial t} + (\nabla\times\vec{B})^i$$

$$= \mu_0 J^i \quad \text{by the Ampere-Maxwell law} \quad \nabla\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

Result $$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \quad \text{where} \quad J^\mu = \begin{bmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{bmatrix}$$

Comments

(1) Two of Maxwell's equations are contained in $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$.

(2) Tensor form $\Rightarrow$ manifestly covariant.

(3) J^μ must transform as a Lorentz vector.

(4) Since $F^{\mu\nu}$ is antisymmetric, $\partial_\mu\partial_\nu F^{\mu\nu} = 0$.

I.e. $\frac{\partial J^\mu}{\partial x^\mu} = 0 \longleftarrow \frac{\partial J^0}{\partial x^0} + \frac{\partial J^i}{\partial x^i} = \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$

THE CONTINUITY EQUATION

UND NUN WIR MUSSEN DER ALTERE ZWEI VON MAXWELL GLEICHUNGEN

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Tensor Analysis — the totally antisymmetric tensor

- In 3 dimensions, it's the Levi-Civita tensor

$$\epsilon_{ijk} = \begin{cases} +1 & \text{for } ijk = 123,\ 231,\ 312 \\ -1 & \text{for } ijk = 213,\ 132,\ 321 \\ 0 & \text{otherwise (i.e., if } ijk \text{ are not all different)} \end{cases}$$

Note that ϵ_{ijk} is totally antisymmetric.

E.g., $\epsilon_{jik} = -\epsilon_{ijk}$; or $\epsilon_{kji} = -\epsilon_{ijk}$; etc.

- In 3 dimensions, we can construct a vector from a tensor by contraction of indices with ϵ_{ijk}

$$A_i = \epsilon_{ijk} T_{jk} \qquad \left(\sum_{jk} \text{ implied}\right) \qquad \begin{cases} A_x = T_{yz} - T_{zy} \\ A_y = T_{zx} - T_{xz} \\ A_z = T_{xy} - T_{yx} \end{cases}$$

Example The cross product of vectors : $\vec{C} = \vec{A} \times \vec{B}$

$$\underset{\text{vector}}{C_i} = \epsilon_{ijk} \underbrace{A_j B_k}_{\text{tensor}} \qquad \begin{cases} C_x = A_y B_z - A_z B_y \\ C_y = A_z B_x - A_x B_z \\ C_z = A_x B_y - A_y B_x \end{cases}$$

- In 3 dimensions, the curl

$$\underset{\text{vector}}{(\nabla \times \vec{F})_i} = \epsilon_{ijk} \underbrace{\frac{\partial}{\partial x_j} F_k}_{\text{tensor}}$$

All that is for 3 dimensions.

Now generalize it to 4 dimensional space time.

The totally antisymmetric tensor in 4 dimensions 12.4/5

$$\epsilon_{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{if } \mu\nu\alpha\beta = 0123 \text{ or any even permutation of } 0123 \\ -1 & \text{if } \mu\nu\alpha\beta = 1023 \text{ or any odd permutation of } 0123 \\ 0 & \text{otherwise [i.e., if } \mu\nu\alpha\beta \text{ are not all different]} \end{cases}$$

Properties

$$\epsilon_{\nu\mu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta}\,, \quad \text{etc.}$$

$$\epsilon_{2031} = -\epsilon_{0231} = -\epsilon_{0123} = -1\,, \quad \text{etc.}$$

- How to construct the DUAL TENSOR of $F^{\mu\nu}$:

$$G_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$$

$\sum_{\alpha,\beta}$ of course

Lower indices tensor

$$G^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}G_{\rho\sigma}$$

raise indices with $g^{\mu\nu}$

$g^{\mu\nu} = \text{diag}(-1,1,1,1)$

i.e., same as $g_{\mu\nu}$.

$$G_{\mu\nu} = g_{\mu\rho}g_{\nu\sigma}G^{\rho\sigma}$$

lower indices with $g_{\mu\nu}$

Crucial point is that $G^{\mu\nu}$ is a tensor;

$G'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}G^{\alpha\beta}$ is the Lorentz transformation.

The DUAL TENSOR

$$G_{\mu\nu} = 0 \quad \text{if } \mu = \nu$$

$$G_{0i} = \tfrac{1}{2}\epsilon_{0i\alpha\beta}F^{\alpha\beta} = \tfrac{1}{2}\epsilon_{ijk}F^{jk} = \tfrac{1}{2}\overbrace{\epsilon_{ijk}\epsilon_{jkl}}^{2\delta_{il}}B^{l} = B^{i}$$

$$G^{0i} = -G_{0i} = -B^{i}$$

$$G^{i0} = +B^{i}$$

$$G_{ij} = \tfrac{1}{2}\epsilon_{ij\alpha\beta}F^{\alpha\beta} = \tfrac{1}{2}\epsilon_{ij0k}F^{0k} + \tfrac{1}{2}\epsilon_{ijk0}F^{k0}$$

$$= \epsilon_{ijk}F^{0k} = \epsilon_{ijk}\frac{E^{k}}{c}$$

$$G^{ij} = \epsilon_{ijk}\frac{E^{k}}{c}$$

Put all that together in a 4x4 matrix: 12.4/6

$$G^{\mu\nu} = \begin{array}{c|cccc} \mu \backslash \nu & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & -B_x & -B_y & -B_z \\ 1 & B_x & 0 & E_z/c & -E_y/c \\ 2 & B_y & -E_z/c & 0 & E_x/c \\ 3 & B_z & \frac{E_y}{c} & -\frac{E_x}{c} & 0 \end{array}$$

Compare it to $F^{\mu\nu}$.

$$G^{\mu\nu}(\vec{E},\vec{B}) = F^{\mu\nu}(-\vec{B},\vec{E})$$

i.e., reversal of $\vec{E}$ and $\vec{B}$.

Replacing $\frac{\vec{E}}{c} \to -\vec{B}$ and $\vec{B} \to \frac{\vec{E}}{c}$ converts $F^{\mu\nu} \to G^{\mu\nu}$.

<u>The field equations</u> Evaluate $\frac{\partial G^{\mu\nu}}{\partial x^\nu}$.

$\boxed{\text{Case } \mu = 0}$ $$\frac{\partial G^{0\nu}}{\partial x^\nu} = \frac{\partial G^{0i}}{\partial x^i} = \nabla\cdot(-\vec{B}) = -\nabla\cdot\vec{B} = 0$$

by Gauss's law; $\nabla\cdot\vec{B} = 0$.

$\boxed{\text{Case } \mu = i}$ $$\frac{\partial G^{i\nu}}{\partial x^\nu} = \frac{\partial G^{i0}}{\partial x^0} + \frac{\partial G^{ij}}{\partial x^j}$$

$$= \frac{1}{c}\frac{\partial}{\partial t} B^i + \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{E^k}{c}$$

$$= \frac{1}{c}\left\{\frac{\partial \vec{B}}{\partial t} + \nabla\times\vec{E}\right\}^i = 0$$

by Faraday's law;
$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$.

So, these two Maxwell equations are simply

$$\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \quad \text{where} \quad G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

Maxwell's equations in tensor form 12.4/7

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \quad \text{where} \quad J^\mu = \begin{bmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{bmatrix}$$

$$\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \quad \text{where} \quad G_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

Quiz Question

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(A) $F^{\mu\nu} F_{\mu\nu}$ is a <u>*scalar*</u>, i.e., invariant with respect to Lorentz transformations.
Express $F^{\mu\nu} F_{\mu\nu}$ in terms of **E** and **B**.

(B) $F^{\mu\nu} G_{\mu\nu}$ is a <u>*scalar*</u>, i.e., invariant with respect to Lorentz transformations.
Express $F^{\mu\nu} G_{\mu\nu}$ in terms of **E** and **B**.