The Electromagnetic Field Tensor

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

Quiz Question

(A) $F^{\mu\nu} F_{\mu\nu}$ is a *scalar*, i.e., invariant with respect to Lorentz transformations.

Express $F^{\mu\nu} F_{\mu\nu}$ in terms of **E** and **B**.

(B) $F^{\mu\nu} G_{\mu\nu}$ is a *scalar*, i.e., invariant with respect to Lorentz transformations.

Express $F^{\mu\nu} G_{\mu\nu}$ in terms of **E** and **B**.

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

Calculate $F^{\mu\nu} F_{\mu\nu}$

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

$-E^2/c^2$

$$F^{\mu\nu} F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \textbf{ DOT } \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

$-E^2/c^2$

B^2+B^2

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FIELD TRANSFORMATIONS

Consider two inertial frames with these coordinates —

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & +B_y & -B_x & 0 \end{bmatrix}$$

$(\mu, \nu = 0, 1, 2, 3)$

$\mathcal{F}: (x^0, x^1, x^2, x^3) = (ct, x, y, z)$

$\mathcal{F}': (x'^0, x'^1, x'^2, x'^3) = (ct', x', y', z')$

The Lorentz transformation from $\mathcal{F}$ to $\mathcal{F}'$ is

$$x'^\mu = \Lambda^\mu{}_\lambda x^\lambda \qquad \left(\sum_{\lambda=0}^{3} \text{ is implied}\right)$$

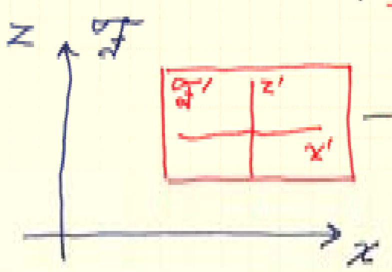
The electromagnetic field tensor must transform as tensor

$$\underbrace{F'^{\mu\nu}(x')}_{} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \underbrace{F^{\rho\sigma}(x)}_{}$$

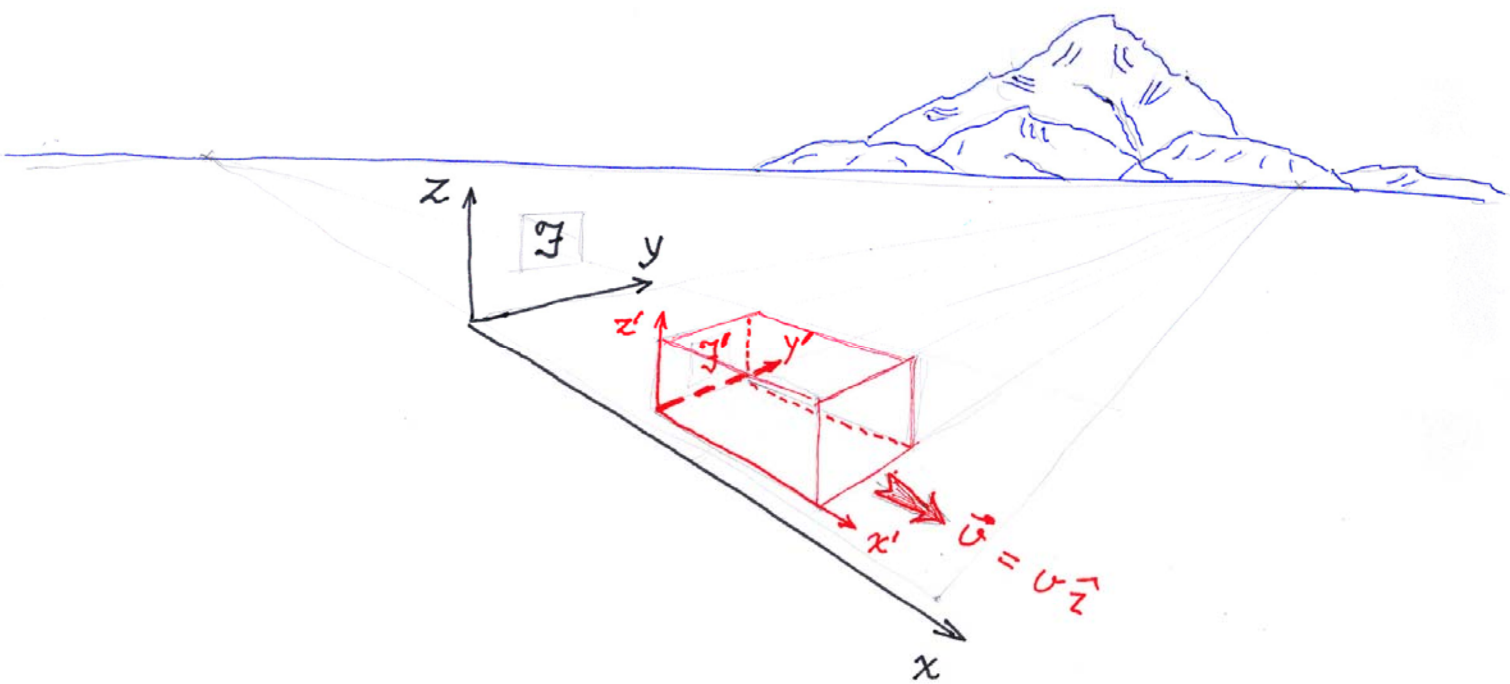
Given $\vec{E}(x)$ and $\vec{B}(x)$ observed in frame $\mathcal{F}$...

... what are $\vec{E}'(x')$ and $\vec{B}'(x')$ observed in frame $\mathcal{F}'$?

First, suppose $\mathcal{F}'$ moves with velocity $\vec{v} = v\hat{x}$ relative to $\mathcal{F}$. (Later, generalize to an arbitrary direction).



$$\Lambda^\mu{}_\nu = \begin{array}{c|c} & \nu \rightarrow \;\; 0 \quad 1 \quad 2 \quad 3 \\ \hline \begin{array}{c} \mu\downarrow \\ 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$


z
$\mathcal{F}$
y
z'
$\mathcal{F}'$
y'
x'
$\vec{v} = v\hat{x}$
x

The electric field in $\mathcal{F}'$

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- $E'_x = c F'^{01} = c \Lambda^0{}_\rho \Lambda^1{}_\sigma F^{\rho\sigma}$ $\quad \rho = 0,1 \quad \sigma = 0,1 \quad \rho \neq \sigma$

$$E'_x = c\, \Lambda^0{}_0 \Lambda^1{}_1 F^{01} + c\, \Lambda^0{}_1 \Lambda^1{}_0 F^{10}$$

$$= c\, \gamma\, \gamma\, \frac{E_x}{c} + c\,(-\beta\gamma)(-\beta\gamma)\left(\frac{-E_x}{c}\right)$$

$$= E_x \qquad \boxed{\gamma^2 (1-\beta^2) = 1} \qquad \beta = v/c \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

- $E'_y = c F'^{02} = c \Lambda^0{}_\rho \Lambda^2{}_\sigma F^{\rho\sigma}$ $\quad \rho = 0,1 \quad \sigma = 2$

$$E'_y = c \Lambda^0{}_0 \Lambda^2{}_2 F^{02} + c \Lambda^0{}_1 \Lambda^2{}_2 F^{12}$$

$$= c \cdot \gamma \cdot 1 \cdot \frac{E_y}{c} + c \cdot (-\beta\gamma) \cdot 1 \cdot B_z$$

$$= \gamma\, (E_y - v B_z)$$

- $E'_z = c F'^{03} =$ similarly $= \gamma\, (E_z + v B_y)$

The transformed electric field

$E'_x = E_x$	$E_x = E'_x$
$E'_y = \gamma\,(E_y - v B_z)$	$E_y = \gamma\,(E'_y + v B'_z)$
$E'_z = \gamma\,(E_z + v B_y)$	$E_z = \gamma\,(E'_z - v B'_y)$
$\mathcal{F}' \leftarrow \mathcal{F}$	$\mathcal{F} \leftarrow \mathcal{F}'$

these field components would be expressed in x'^μ

than in x^μ.

The magnetic field in $\mathcal{F}'$ — 12.5/3

- $$B'_x = F'^{23} = \Lambda^2{}_{\underset{\rho=2}{\rho}}\, \Lambda^3{}_{\underset{\sigma=3}{\sigma}}\, F^{\rho\sigma} = F^{23} = B_x$$

- $$B'_y = F'^{31} = \Lambda^3{}_{\underset{\rho=3}{\rho}}\, \Lambda^1{}_{\underset{\sigma=0,1}{\sigma}}\, F^{\rho\sigma}$$

$$B'_y = \Lambda^1{}_0 F^{30} + \Lambda^1{}_1 F^{31}$$
$$= -\beta\gamma\left(\frac{-E_z}{c}\right) + \gamma B_y$$
$$= \gamma\left(B_y + \frac{v}{c^2} E_z\right)$$

- $$B'_z = F'^{12} = \text{similar } B_y = \gamma\left(B_z - \frac{v}{c^2} E_y\right)$$

The transformed magnetic field

$B'_x = B_x$	$B_x = B'_x$
$B'_y = \gamma\left(B_y + \frac{v}{c^2} E_z\right)$	$B_y = \gamma\left(B'_y - \frac{v}{c^2} E'_z\right)$
$B'_z = \gamma\left(B_z - \frac{v}{c^2} E_y\right)$	$B_z = \gamma\left(B'_z + \frac{v}{c^2} E'_y\right)$

these functions expressed in x'

these in x

All these results assume $\mathcal{F}'$ moves with velocity $v\hat{z}$ relative to $\mathcal{F}$. By inspection we can generalize the results for any direction of $\vec{v}$.

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If frame $\mathcal{F}'$ moves with velocity $\vec{v}$ (in any direction) relative to frame $\mathcal{F}$ —

$$E'_{\parallel} = E_{\parallel} \qquad\qquad E_{\parallel} = E'_{\parallel}$$

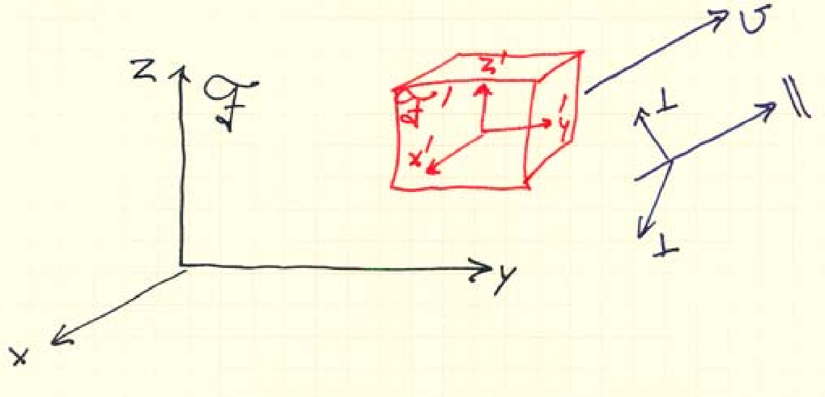
$$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v}\times\vec{B}_{\perp}) \qquad\qquad \vec{E}_{\perp} = \gamma(\vec{E}'_{\perp} - \vec{v}\times\vec{B}'_{\perp})$$

$$B'_{\parallel} = B_{\parallel} \qquad\qquad B_{\parallel} = B'_{\parallel}$$

$$\vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{\vec{v}}{c^2}\times\vec{E}_{\perp}\right) \qquad\qquad \vec{B}_{\perp} = \gamma\left(\vec{B}'_{\perp} + \frac{\vec{v}}{c^2}\times\vec{E}'_{\perp}\right)$$

$\mathcal{F}' \leftarrow \mathcal{F}$ $\qquad\qquad$ $\mathcal{F} \leftarrow \mathcal{F}'$

These are the electric and magnetic field components that are parallel ($\parallel$) to $\vec{v}$ and perpendicular ($\perp$) to $\vec{v}$



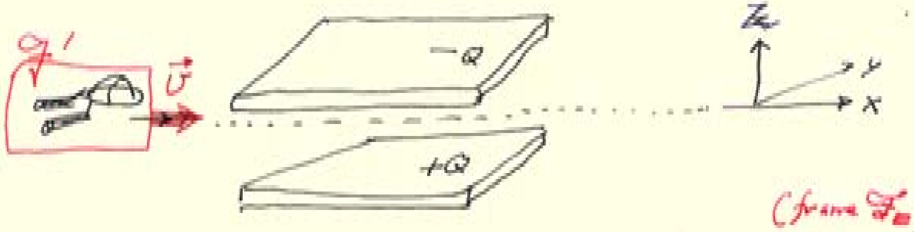
See Table 12.3

Table 12.3: Lorentz transformations of various quantities. The inertial frame $\mathcal{F}'$ moves with velocity $\mathbf{v}$ with respect to frame $\mathcal{F}$. The components denoted $\parallel$ and $\perp$ are parallel and perpendicular to $\mathbf{v}$.

coordinates	
$t' = \gamma(t - vx_\parallel/c^2)$	$t = \gamma(t' + vx'_\parallel/c^2)$
$x'_\parallel = \gamma(x_\parallel - vt)$	$x_\parallel = \gamma(x'_\parallel + vt')$
$\mathbf{x}'_\perp = \mathbf{x}_\perp$	$\mathbf{x}_\perp = \mathbf{x}'_\perp$
energy and momentum	
$E' = \gamma(E - vp_\parallel)$	$E = \gamma(E' + vp'_\parallel)$
$p'_\parallel = \gamma(p_\parallel - vE/c^2)$	$p_\parallel = \gamma(p'_\parallel + vE'/c^2)$
$\mathbf{p}'_\perp = \mathbf{p}_\perp$	$\mathbf{p}_\perp = \mathbf{p}'_\perp$
velocity	
$u'_\parallel = (u_\parallel - v)/(1 - vu_\parallel/c^2)$	$u_\parallel = (u'_\parallel + v)/(1 + vu'_\parallel/c^2)$
$\mathbf{u}'_\perp = (1/\gamma)\mathbf{u}_\perp/(1 - vu_\parallel/c^2)$	$\mathbf{u}_\perp = (1/\gamma)\mathbf{u}'_\perp/(1 + vu'_\parallel/c^2)$
electric and magnetic fields	
$E'_\parallel = E_\parallel$	$E_\parallel = E'_\parallel$
$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}_\perp)$	$\mathbf{E}_\perp = \gamma(\mathbf{E}'_\perp - \mathbf{v} \times \mathbf{B}'_\perp)$
$B'_\parallel = B_\parallel$	$B_\parallel = B'_\parallel$
$\mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \mathbf{v} \times \mathbf{E}_\perp/c^2)$	$\mathbf{B}_\perp = \gamma(\mathbf{B}'_\perp + \mathbf{v} \times \mathbf{E}'_\perp/c^2)$

Example 12.5/5

The Starship Enterprise has encountered a large parallel plate capacitor in space. The charges on the plates are $+Q$ and $-Q$. In the rest frame of the capacitor (frame $\mathcal{F}_C$) the area is $A\hat{k}$ and the velocity of the Enterprise is $\vec{v} = v\hat{i}$.

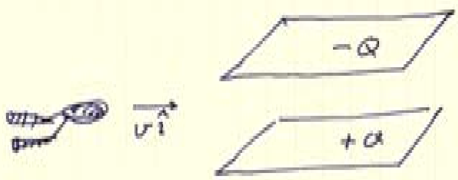


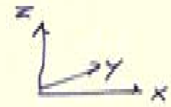
Mr. Spock uses sensors attached to the Enterprise (frame $\mathcal{F}_E$) to measure the electric and magnetic fields as the starship passes between the plates. What are the field measurements that he will obtain?

Frame $\mathcal{F}_C$:
$$(E_x, E_y, E_z) = \left(0, 0, \frac{Q}{\epsilon_0 A}\right)$$
$$(B_x, B_y, B_z) = (0, 0, 0)$$

Frame $\mathcal{F}_E$:
$$(E'_x, E'_y, E'_z) = \left(E_x, \gamma(E_y - vB_z), \gamma(E_z + vB_y)\right) = \left(0, 0, \frac{\gamma Q}{\epsilon_0 A}\right)$$
$$(B'_x, B'_y, B'_z) = \left(B_x, \gamma\left(B_y + \frac{v}{c^2}E_z\right), \gamma\left(B_z - \frac{v}{c^2}E_y\right)\right) = \left(0, \frac{\gamma v}{c^2}\frac{Q}{\epsilon_0 A}, 0\right)$$

Spock's measurements :
$$\vec{E}' = \frac{\gamma Q}{\epsilon_0 A}\hat{k} \quad \text{and} \quad \vec{B}' = \frac{\gamma v}{c^2}\frac{Q}{\epsilon_0 A}\hat{j}$$





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$$\vec{E}' = \frac{\gamma Q}{\epsilon_0 A}\,\hat{k} \quad \text{and} \quad \vec{B}' = \frac{\gamma v}{c^2}\frac{Q}{\epsilon_0 A}\,\hat{\jmath}$$

$$\vec{B}' = \mu_0 \frac{\gamma v Q}{A}\,\hat{\jmath}$$

Does it make sense?

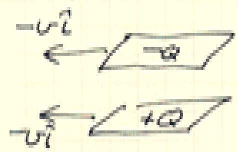
Yes; look at it from the frame of reference $\mathcal{F}_E$; I.e., the frame of reference in which the Enterprise is at rest and the capacitor plates move with velocity $-v\hat{\imath}$.

- The electric field is due to the Lorentz contracted charge density, $\sigma' = \frac{Q}{A'}$ where $\underbrace{A' = \frac{A}{\gamma}}_{\text{Lorentz contracted area} = A'}$

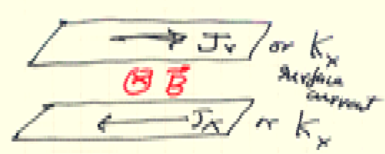
  By Gauss's law,

  $$\therefore\ \vec{E}' = \frac{\gamma Q}{\epsilon_0 A}\,\hat{k}$$

- The magnetic field is due to the 2 current sheets



is like



By Ampère's law, $B_y \ell = \mu_0 K_x \ell$

$$B_y = \mu_0 K_x = \mu_0 \sigma' v = \mu_0 \frac{\gamma Q}{A} v$$

Exercise

(A) Show that $E^2 - c^2 B^2$ is invariant; $\leftarrow \left(\frac{Q}{\epsilon_0 A}\right)^2$

(B) Show that $\vec{E}\cdot\vec{B}$ is invariant $\leftarrow 0$

in either Lorentz frame.

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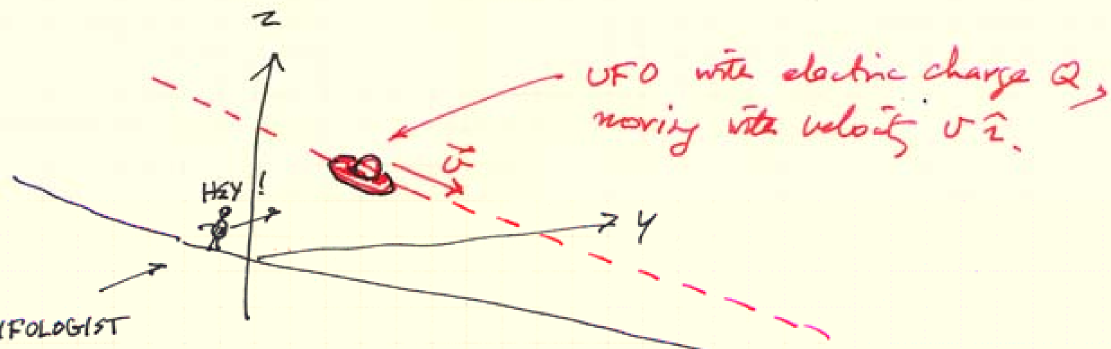
Another Example

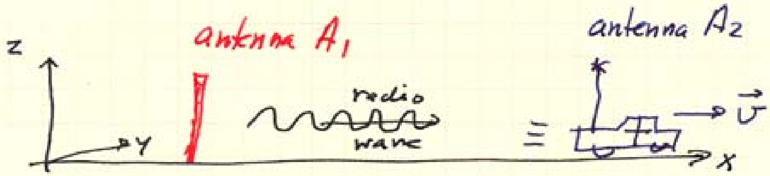
Determine the electric and magnetic field due to a charged particle that moves with velocity $\vec{v}$.

(NEXT TIME)



What $\vec{E}$ and $\vec{B}$ would be observed by the ufologist?

Another example 12.5/8



- A_1 is the transmitter. In frame $\mathcal{F}$, A_1 is at rest and the electromagnetic wave fields are

$$\vec{E}(\vec{x},t) = E_0 \sin(kx-\omega t)\,\hat{k}$$
$$\vec{B}(\vec{x},t) = -\frac{E_0}{c} \sin(kx-\omega t)\,\hat{j}$$

- A_2 is the receiver. A_2 moves with velocity $v\,\hat{i}$ w.r.t. the frame $\mathcal{F}$. {Frame $\mathcal{F}'$ is the rest frame of A_2.}

(A) Determine the frequency of the radio wave observed by A_2. [f_2, depends on f_1 and v.]

(B) Determine the amplitude of oscillation of the electric field observed by A_2. [$E_0^{(2)}$ depends on E_0 and v.]

Quiz Question: Answer (A) and (B).