The Electromagnetic Field Tensor

\[
F^{\mu\nu} = \begin{bmatrix}
0 & E_x / c & E_y / c & E_z / c \\
-E_x / c & 0 & B_z & -B_y \\
-E_y / c & -B_z & 0 & B_x \\
-E_z / c & B_y & -B_x & 0
\end{bmatrix}
\]

Table 12.3: Lorentz transformations of various quantities. The inertial frame \( \mathcal{F}' \) moves with velocity \( \mathbf{v} \) with respect to frame \( \mathcal{F} \). The components denoted \( \parallel \) and \( \perp \) are parallel and perpendicular to \( \mathbf{v} \).

<table>
<thead>
<tr>
<th>coordinates</th>
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</thead>
<tbody>
<tr>
<td>( t' = \gamma(t - \mathbf{v} x_\parallel / c^2) )</td>
</tr>
<tr>
<td>( x'<em>\parallel = \gamma(x</em>\parallel - \mathbf{v} t) )</td>
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<tr>
<td>( x'<em>\perp = x</em>\perp )</td>
</tr>
<tr>
<td>( t = \gamma(t' + \mathbf{v} x'_\parallel / c^2) )</td>
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<tr>
<td>( x_\parallel = \gamma(x'_\parallel + \mathbf{v} t') )</td>
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<td>( x_\perp = x'_\perp )</td>
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<th>energy and momentum</th>
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<td>( E' = \gamma(E - \mathbf{v} p_\parallel) )</td>
</tr>
<tr>
<td>( p'<em>\parallel = \gamma(p</em>\parallel - \mathbf{v} E / c^2) )</td>
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<tr>
<td>( p'<em>\perp = p</em>\perp )</td>
</tr>
<tr>
<td>( E = \gamma(E' + \mathbf{v} p'_\parallel) )</td>
</tr>
<tr>
<td>( p_\parallel = \gamma(p'_\parallel + \mathbf{v} E' / c^2) )</td>
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<tr>
<td>( u'<em>\parallel = (u</em>\parallel - \mathbf{v})/(1 - \mathbf{v} u_\parallel / c^2) )</td>
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<tr>
<td>( u'<em>\perp = (1/\gamma)u</em>\perp/(1 - \mathbf{v} u_\parallel / c^2) )</td>
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<tr>
<td>( u_\parallel = (u'<em>\parallel + \mathbf{v})/(1 + \mathbf{v} u'</em>\parallel / c^2) )</td>
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<td>( u_\perp = (1/\gamma)u'<em>\perp/(1 + \mathbf{v} u'</em>\parallel / c^2) )</td>
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<td>( E'<em>\parallel = E</em>\parallel )</td>
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<tr>
<td>( E'<em>\perp = \gamma(E</em>\perp + \mathbf{v} \times \mathbf{B}_\perp) )</td>
</tr>
<tr>
<td>( B'<em>\parallel = B</em>\parallel )</td>
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<tr>
<td>( B'<em>\perp = \gamma(B</em>\perp - \mathbf{v} \times E'_\perp / c^2) )</td>
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<td>( E_\parallel = E'_\parallel )</td>
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Electromagnetism and Relativity (6) 12.5

The Fields of a Moving Charge

A very basic question: What are the fields, \( \mathbf{E}(x,t) \) and \( \mathbf{B}(x,t) \), due to a charged particle that moves with constant velocity? 

\[ \mathbf{E} = \mathbf{E}^\prime \]

Suppose the charge of mass with velocity \( \mathbf{v} = \mathbf{v}_0 \) on the x-axis of the inertial frame \( \mathcal{F} \).

The position \( \mathbf{q} \) is \( \mathbf{q}^\prime = \mathbf{v}_0 t \).

The problem is to determine \( \mathbf{E}(x,t) \) and \( \mathbf{B}(x,t) \).

We'll find the answer by applying the Lorentz transformations.

* Let \( \mathcal{F}' \) be the rest frame of \( \mathcal{F} \), in which \( \mathbf{q}^\prime = \mathbf{0} \) at rest at the origin. Then \( \mathbf{q}_0 = 0 \).

In this frame, the fields are

\[ \mathbf{E}'(\mathbf{r}^\prime) = \frac{\mathbf{q}^\prime}{4\pi\varepsilon_0 (\mathbf{v}^\prime)^2} \quad \text{and} \quad \mathbf{B}'(\mathbf{r}^\prime) = 0. \]

(Contact field)
Locally transforming of coordinates

\[ x'^u = \Lambda'^u \nu \ x'^\nu \]
where

\[ \Lambda'^u \nu = \begin{bmatrix} \gamma & \beta \gamma & 0 & 0 \\ \beta \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

- \[ x^0 = \Lambda^0 \nu \ x'^\nu = \Lambda^0 \nu \ x'^0 + \Lambda^0 \nu \ x'^1 \]
  \[ = \gamma x^0 + \beta \gamma x^1 \]
  \[ t = \gamma (t' + \frac{\nu}{c^2} x') \]
  \( (x' = 0) \)
The time coordinate of the particle is \( t = \gamma t' \);
The time dilation: \( \Delta t = \gamma \Delta t' > \Delta t' \).

- \[ x^1 = \Lambda^1 \nu \ x'^\nu = \Lambda^1 \nu \ x'^0 + \Lambda^1 \nu \ x'^1 \]
  \[ = \beta \gamma ct' + \gamma x^1 \]
  \( (x' = 0) \)
The \( x \) coordinate of the particle is
  \[ x = \gamma (x^1 + \nu t) \]
  \( (x' = 0) \)

The particle moves on the \( x \) axis with velocity \( \nu \).

- \[ y = y' \beta \gamma \quad \text{and} \quad z = z' \gamma \]

The particle has \( \vec{v} = (\nu t', 0, 0) \).

- Note that

\[ | \vec{v} | = \sqrt{(x^1)^2 + (\nu t)^2 + (z')^2} \]
  \[ = \sqrt{\gamma^2 (x - \nu t)^2 + \nu^2 + z'^2} \]

Check:

\[ \gamma (x - \nu t) = \gamma \cdot \gamma (x^1 + \nu t') - \gamma \nu \gamma (t' + \frac{\nu}{c^2} x') \]
  \[ = \gamma^2 (1 - \beta^2) x' = x' \]
Lecture 12.5

Lecture 12.6

\[ F^{\mu \nu} = \Lambda^{\alpha \beta} \Lambda^{\gamma \delta} F_{\alpha \beta} \]

The Electric field

\[ E_{i}' = c F_{\alpha i} = c \Lambda^{\alpha \beta} \Lambda^{\gamma i} F_{\beta \gamma} \]

I could go on, but I already did this last time.

\[ E_{\parallel} = E_{\parallel}' \]

\[ E_x = E_x' = \frac{q}{4\pi \varepsilon_0} \frac{x'}{12x'^3} = \frac{q}{4\pi \varepsilon_0} \frac{y(x-vt)}{D^3} \]

Let \( D = \sqrt{y^2(x-vt)^2 + y^2 + z^2} \)

\[ E_y = y \left( \frac{q}{4\pi \varepsilon_0} \frac{x'}{12x'^3} \right) = \frac{q y y}{4\pi \varepsilon_0 D^3} \]

\[ E_z = y \left( \frac{q}{4\pi \varepsilon_0} \frac{x'}{12x'^3} \right) = \frac{q y z}{4\pi \varepsilon_0 D^3} \]

\[ E^{(3+1)} = \frac{q y}{4\pi \varepsilon_0} \frac{(x-vt)^2 + y^2 + z^2}{D^3} \]

Lec 12.5 Lecture 12.5
\[ E(x, t) = \frac{q_x}{4\pi\varepsilon_0} \frac{(x-vt)^2 + y^2 + z^2}{D^3} \]

\[ D = \sqrt{x^2(x-vt)^2 + y^2 + z^2} \]

**Comments**

1. If \( v = 0 \), the \( E \) is the Coulomb field.

2. The direction of \( E(x, t) \) is radially away from the instantaneous position of \( q_x \).

3. The magnitude of \( E \) for points in the \( x \) axis is

\[ E(x, \infty) = \frac{q_x}{4\pi\varepsilon_0} \frac{x-vt}{x^2} = \frac{1}{\varepsilon_0} \frac{q_x}{2\pi d^2} \]

where \( d \) is the distance from \( x \) to \( q_x \), i.e., \( d = x-vt \).

   The magnitude is less than the Coulomb field by the factor \( \frac{1}{\sqrt{2}} \). (points on \( x \) axis)

4. The magnitude of \( E \) for points in the plane \( x = y = vt \)

\[ E(vt, y, z) = \frac{q_x}{4\pi\varepsilon_0} \frac{\sqrt{y^2 + z^2}}{(y^2 + z^2)^{3/2}} = \frac{q_x}{4\pi\varepsilon_0 d^2} \]

where \( d \) is the distance to the charge \( q_x \), i.e., \( d = \sqrt{y^2 + z^2} \).

   This magnitude is greater than the Coulomb field by the factor \( \sqrt{2} \). (points on the plane \( x = y \))
The Magnetic Field

\[ B_x = F^{1.5} = N I B \]

\[ B_x = B_x' = 0 \]

x-component

y and z components

\[ B'_1 = y \left( \frac{x' B'_y + y' x' B'_z}{c^2} \right) \]

\[ B'_1 = \frac{y V}{c^2} \left( \frac{c}{c} \right) \]

\[ B'_1 = \frac{y V}{c^2} \frac{y'}{m_c} \frac{x' B'_x + y' B'_z}{c^2} \]

\[ B'_1 = \frac{\mu_0}{4\pi} \frac{y V}{D^2} \left( -z' + \frac{y'}{c^2} \right) \]

Comments

(1) The direction of \( \vec{B} \) is azimuthally around the x-axis.

(2) On the \( x \)-axis, \( \vec{B} = 0 \).

(3) On the plane \( x = y \) (\( \vec{x}_y \)), \( \vec{B} = \frac{\mu_0}{4\pi} \frac{y V}{D^2} \vec{y} \).

(4) On the plane \( x = z \),

\[ \frac{B}{E} = \frac{\mu_0}{4\pi} \frac{y V}{D^2} = \frac{y}{c^2} \frac{x}{m_c + \frac{1}{2}} \]

(Example: For an e.m. wave, \( B/E = k \)).
Quiz Question
A UFO flies over head. It has electric charge $Q = 10,000$ coulombs. It moves with speed $0.9c$ parallel to the x axis, at height $1,000$ m above the ground. The ufologist on the ground is ready for it, and measures the electric and magnetic fields. Calculate the field measurements when the UFO is directly above the ufologist.

(A) Electric field     (B) Magnetic field