Motion of one planet and a star—25 Feb

Announcements:

- Monday: Missouri (Ask Me State) Club.
- Homework 5 not accepted after today. Answers will be on angel after class.
- Send me equations to put on cheat sheet before 8:00am, Fri, 4 March.
- Friday: Midterm test. Covers topics through comet tails (first part of 18 Feb). Does not cover Pluto and Kuiper Belt (last part of 18 Feb)

Outline:

- Motion of 2 bodies can be changed into the motion of the center of mass and the orbit of a single body.
- Kepler's Laws

The problem

A planet and the sun are in orbit. There are no other bodies.

m_s at $\vec{r}_s$

m_p at $\vec{r}_p$

The momentum is

$$\vec{P} = m_s \frac{d\,\vec{r}_s}{d\,t} + m_p \frac{d\,\vec{r}_p}{d\,t}$$

is conserved.

Reason: There are no forces acting on the two bodies from the outside. Therefore $\vec{F} = \frac{d\,\vec{P}}{d\,t} = 0$.

1. Make up a case where the sun emits radiation and momentum is not conserved.
2. Make up a case where the sun emits radiation and momentun is conserved.

Consider only cases where momentum is conserved.

Choose the center of mass to be stationary

Write

$$\vec{P} = M \frac{d}{dt} \left(m_s \vec{r}_s + m_p \vec{r}_p\right) / M$$, where the total mass $M = (m_s + m_p)$.

Define the center of mass position

$$\vec{R} = \left(m_s \vec{r}_s + m_p \vec{r}_p\right) / \left(m_s + m_p\right)$$

The center of mass position moves at constant speed. No additional information.

Change to a frame where the center of mass is at the origin. $\vec{R} = 0$.

Let

$$\vec{r} = \vec{r}_p - \vec{r}_s$$

Define the reduced mass μ by

$\mu = m_s m_p / (m_s + m_p)$ or $\frac{1}{\mu} = \frac{1}{m_s} + \frac{1}{m_p}$

Then solve

$$m_s \vec{r}_s = -m_p \vec{r}_p = -m_p (\vec{r} + \vec{r}_s)$$

to get

$$\vec{r}_s = -\mu / m_s \vec{r}$$
$$\vec{r}_p = \mu / m_p \vec{r}$$

1. In the center of mass frame, ___ is the vector from the sun to the planet and ___ is the vector from the origin to the planet.

A. $\vec{r}_p, \vec{r}$

B. $\vec{r}, \vec{r}_p$

New equation of motion

The equation of motion

$$m_p \frac{d\vec{v}_p}{dt} = -G\, m_s\, m_p (\vec{r}_p - \vec{r}_s) \big/ \left(\left| \vec{r}_p - \vec{r}_s \right| \right)^3$$

can be written

$$\mu \frac{d\vec{v}}{dt} = -G\, M\, \mu\, \vec{r} \big/ r^3 \text{ where } \vec{v} = \frac{d\,\vec{r}}{dt}$$

We have successfully changed a 2-body problem into a one-body problem. There are two equations of motion. (1) The center of mass moves at constant speed. (2) In the center of mass frame is a new object with mass μ that is pulled by a stationary mass M.

1. Suppose I solve the equation of motion. How do I figure out where the planet is? The position of the planet is ___ $\vec{r}$ from the center of mass.

A. exactly
B. a bit beyond
C. a bit under

2. Suppose two stars of equal mass $m_\odot$ orbit. In the one-body problem, the mass that pulls to cause the acceleration is

A. $\frac{1}{2}\, m_\odot$
B. $m_\odot$
C. $2\, m_\odot$

Conserved quantities

The total energy is

$$E = \frac{1}{2} m_s v_s^2 + \frac{1}{2} m_p v_p^2 - G m_s m_p \Big/ r.$$

Write in terms of r and

$$E = \frac{1}{2} \mu v^2 - G M \mu / r$$

I could have done that directly, since I changed the equation of motion from a 2-body to a one=body problem.

The angular momentum is

$$\vec{L} = \vec{r} \times \vec{p} \text{ where } \vec{p} = \mu \vec{v}$$

Angular momentum is conserved because the force is radial.

$$\frac{d}{\text{dt}} \vec{L} = \left(\frac{d}{d t} \vec{r}\right) \times \vec{p} + \vec{r} \times \frac{d}{d t} \vec{p}$$

The first term is proportional to $\vec{p} \times \vec{p}$. The second term is proportional to $\vec{r} \times \vec{r}$.

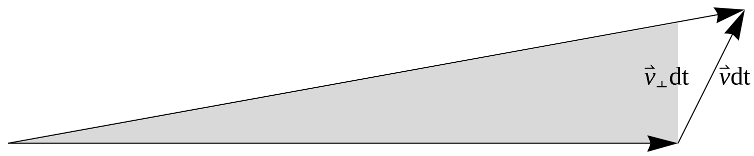
Kepler’s 2nd Law, Law of Equal Areas

Consider $\vec{L}/\mu = \vec{r}\times\vec{v} = \vec{r} \times \frac{d\vec{r}}{dt}$.

$$L/(2\,\mu) = \frac{1}{2}\, r\, \frac{d r_{\perp}}{dt}$$

This is the area swept out in dt. Since the angular momentum is conserved, the area swept out per unit time does not change.

```
p1 = {1, 0}; p2 = {1.1, .2};
Graphics[{LightGray, Polygon[{{0, 0}, p1, p1 + {0, .18}}],
  Black, Arrow[{{0, 0}, p1}], Arrow[{{0, 0}, p2}], Arrow@{p1, p2},
  Text[Style["v⃗dt", Medium], (p1 + p2) / 2, {-1.5, 0}],
  Text[Style["v⃗⊥dt", Medium], p1 + {0, .1}, {.5, 0}]}]
```



- **Make plot**

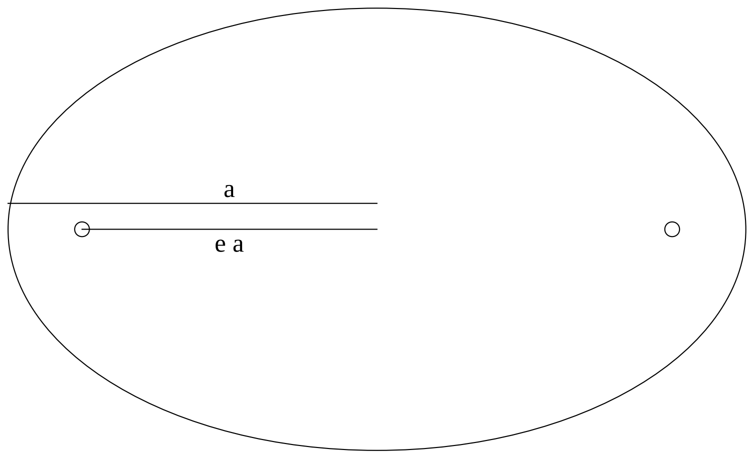
Results

Kepler’s First Law The orbit of a planet is an ellipse with the sun at one focus.

The definition of an ellipse: The sum of the distance between a point and the foci is a constant.

The semimajor axis is a.

The eccentricity is e.



Kepler’s 3rd Law

$$P^2 = 4\pi^2 a^3 / (G M)$$

Energy $\mathcal{E}$ and angular momentum $\mathcal{L}$ or the values per unit mass E and L (called specific energy and angular momentum)

$$L = \left[G M a \left(1 - e^2\right)\right]^{1/2}$$

$$E = -\frac{1}{2} G M / a$$

1. A comet (which has high eccentricity) and a planet (which has low eccentricity) have the same period. S1: The specific energy of the comet is smaller. S2: the specific angular momentun of the comet is less.

A. Both true.

B. S1 is true. S2 is false.

C. S1 is false. S2 is true.

D. Both false.