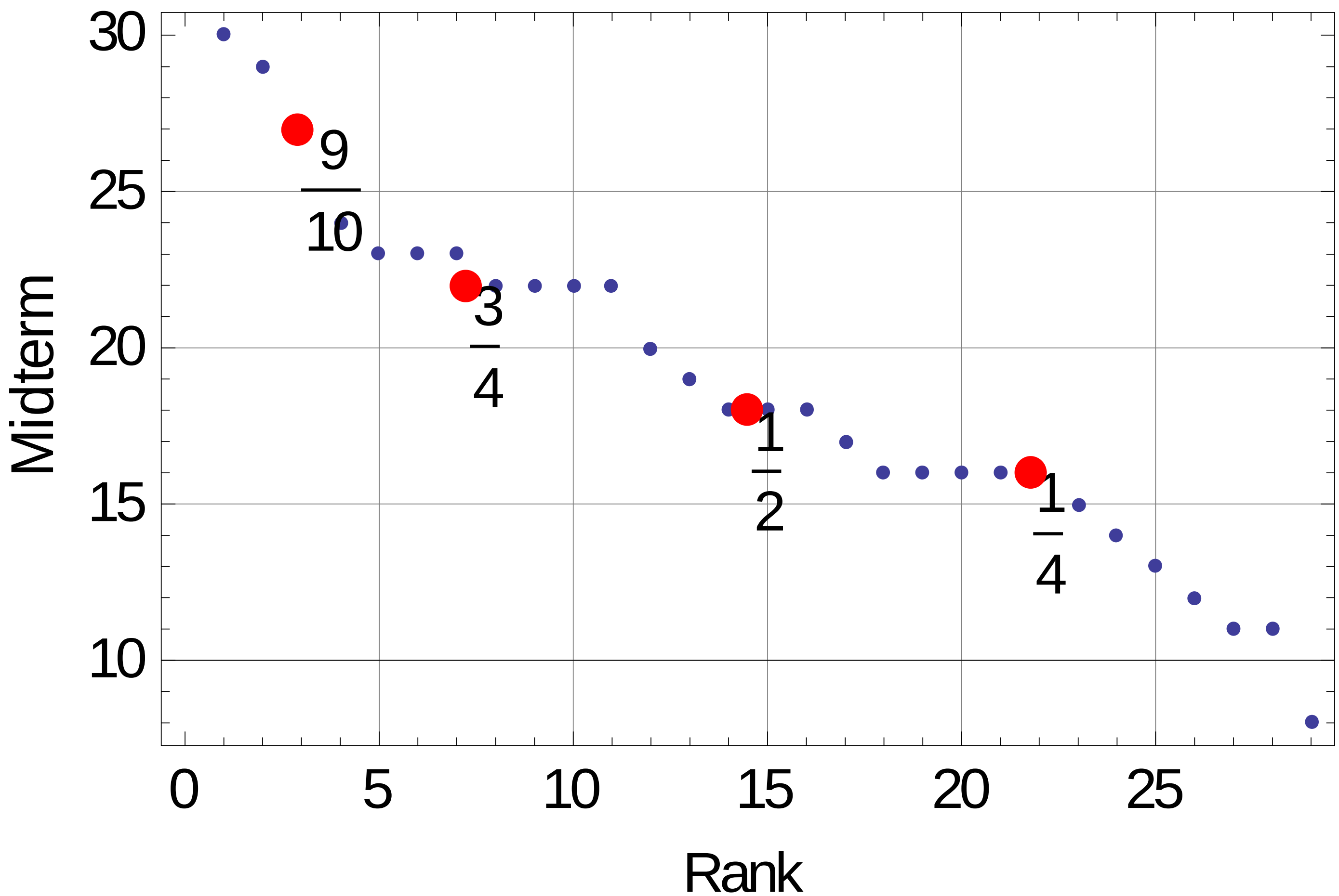
Midterm exam—14 March

- Statistics
  - Bottom quartile 16
  - Median 18
  - Top quartile 22
- Using the Midterm as an indicator
  - Course grade will be curved to make average about 3.0.
  - The grade depends on only 5 problems. 7 points/problem
  - If you are in the bottom tenth (12pts or fewer), you must change the way you are thinking about Ast208.

Midterm grades and rank in class



Suppose you got 20 on the midterm. This chart says your rank is 12th out of 29 students, and you are between ¾ and ½ of the class.

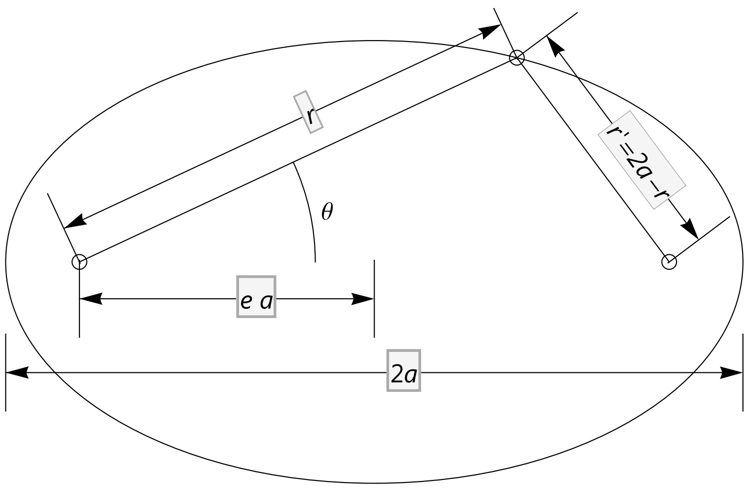
Kepler’s Laws—14 Mar

Outline:

- Motion of 2 bodies can be changed into the motion of the center of mass and the orbit of a single body. (Done before break.)
- Derive Kepler’s 2nd Law, the law of equal areas. (Done before the break.)
- Derive Kepler’s 1st and 3rd Laws

Q: Write Kepler’s First Law on paper and turn it in.

Derivation of Kepler’s First Law, that orbits are ellipses. Equation of an ellipse.



Definition of an ellipse: The sum of the distances from the two foci is a constant.

$r + r' = 2\,a$.

For this derivation, let $a = 1$. Then I don’t have to write a a in a lot of places.
Q: How do I make the answer apply to the case of arbitrary a?

The distance from the planet to the second focus is related to r and θ by

$r'^2 = (r \sin\theta)^2 + (2\,e - r\cos\theta)^2$

$= 4\,e^2 - 4\,e\,r\cos\theta + r^2$.

Since $r + r' = 2$,

$r'^2 = (2 - r)^2 = 4 - 4\,r + r^2$

Combine to get (I added a back.)

$$\boxed{r = a\left(1 - e^2\right)\big/(1 - e\cos\theta)}$$

Derivation of Kepler’s First Law, that the path is an ellipse

Angular momentum is

$\vec{L} = \vec{r}\times\vec{v}$

$= \vec{r}\times\vec{v}_\perp$

since the cross product vanishes for two parallel vectors. (I use specific angular momentum, the angular momentum per unit mass, because the path does not depend on mass.)

Note $\vec{v} = \frac{dr}{dt}\,\hat{r} + r\,\frac{d\hat{r}}{dt}$. Therefore $\vec{v}_\perp = r\,\frac{d\hat{r}}{dt}$.

The direction is out of the plane of the orbit.

Cross the acceleration

$\vec{a} = -G\,M\,\vec{r}\big/r^3$

with $\vec{L}$ to get

$$\vec{a}\times\vec{L} = -G\,M \big/ r^3\, \vec{r}\times\left(\vec{r}\times\vec{v}_\perp\right)$$

Use the right-hand rule to get $\vec{r}\times\left(\vec{r}\times\vec{v}_\perp\right) = -r^2\ \vec{v}_\perp = r^3\ \frac{d\hat{r}}{dt}$.

Stopped here.

$$\frac{d}{dt}\left(\vec{v}\times\vec{L}\right) = G\,M\ \frac{d\hat{r}}{dt}$$

Q: $\vec{a}\times\vec{L} = \frac{d}{dt}\left(\vec{v}\times\vec{L}\right)$ because

A. $\vec{v}$ and $\vec{L}$ are parallel.

B. $\vec{v}$ and $\vec{L}$ are perpendicular.

C. $\vec{L}$ is a constant.

Integrate to get

$$\vec{v}\times\vec{L} = G\,M\ \hat{r} + \vec{D}.$$

$\vec{D}$ is a constant of integration. Define angles from the direction of $\vec{D}$.

Dot with $\vec{r}$ to get $\vec{r}\cdot\left(\vec{v}\times\vec{L}\right) = \vec{r}\times\vec{v}\cdot\vec{L} = L^2$

$$\boxed{L^2 = G\,M\,r + D\,r\cos\theta}$$

$$r\,(1 + D/(G\,M)\cos\theta) = L^2 \big/ (G\,M)$$

This is the equation of an ellipse. Identify

$$e = D/(G\,M)$$

$$\boxed{L^2 = G\,M\,a\left(1 - e^2\right)}$$

Q: Consider an orbit with a low angular momentum. In PHY183, you learned that angular momentum is $L/m = v\,(\text{moment arm})$. At aphelion (farthest from the sun), the angular momentum is low because

A. the moment arm is small.

B. the velocity is small.

C. both the velocity and moment arm are small.

D. the velocity is big.

Q: Consider an orbit with a low angular momentum. When the planet is halfway between perihelion and aphelion, the angular momentum is low because ... (Same foils.)

Energy at perihelion

$$E = \frac{1}{2}\ v_p^2 - G\,M \big/ r_p$$

$$r_p\,E = \frac{1}{2}\ L^2 \big/ r_p - G\,M = \frac{1}{2}\ \frac{L^2}{a\,(1-e)} - G\,M$$

Add to energy at aphelion

$$\left(r_p + r_a\right)E = \frac{1}{2}\ \frac{L^2}{a}\ \frac{2}{1-e^2} - 2\,G\,M$$

$$E = \frac{1}{2}\ \frac{L^2}{a^2\left(1-e^2\right)} - G\,M/a$$

Substitute to get

$$E = \frac{1}{2}\ G\,M/a - G\,M/a$$

$$\boxed{E = -\frac{1}{2}\ G\,M/a}$$

Q: State this in the simplest possible way, so that you will remember it.