Kepler’s 1st & 3rd Law—18 Mar

- Announcements
  - Homework 6 is due Fri, 25 Mar.
- Outline
  - Finish derivation of Kepler’s 1st Law
  - Derive K’s 3rd Law
  - Unbound orbits
  - Finding the position as a function of time

Derivation of Kepler’s First Law, that the path is an ellipse

Angular momentum is

$$\vec{L} = \vec{r}\times\vec{v}$$
$$= \vec{r}\times\vec{v}_\perp$$

since the cross product vanishes for two parallel vectors. (I use specific angular momentum, the angular momentum per unit mass, because the path does not depend on mass.)

Note $\vec{v} = \frac{dr}{dt}\,\hat{r} + r\,\frac{d\hat{r}}{dt}$. Therefore $\vec{v}_\perp = r\,\frac{d\hat{r}}{dt}$.

The direction is out of the plane of the orbit.

Cross the acceleration

$$\vec{a} = -G\,M\,\vec{r}\big/r^3$$

with $\vec{L}$ to get

$$\vec{a}\times\vec{L} = -G\,M\big/r^3\,\vec{r}\times\left(\vec{r}\times\vec{v}_\perp\right)$$

Use the right-hand rule to get $\vec{r}\times\left(\vec{r}\times\vec{v}_\perp\right) = -r^2\,\vec{v}_\perp = r^3\,\frac{d\hat{r}}{dt}$.

$$\frac{d}{dt}\left(\vec{v}\times\vec{L}\right) = G\,M\,\frac{d\hat{r}}{dt}$$

Q: $\vec{a}\times\vec{L} = \frac{d}{dt}\left(\vec{v}\times\vec{L}\right)$ because

A. $\vec{v}$ and $\vec{L}$ are parallel.

B. $\vec{v}$ and $\vec{L}$ are perpendicular.

C. $\vec{L}$ is a constant.

Integrate to get

$$\vec{v}\times\vec{L} = G\,M\,\hat{r} + \vec{D}.$$

$\vec{D}$ is a constant of integration. Define angles from the direction of $\vec{D}$.

Dot with $\vec{r}$ to get $\vec{r}\cdot\left(\vec{v}\times\vec{L}\right) = \vec{r}\times\vec{v}\cdot\vec{L} = L^2$

$$\boxed{L^2 = G\,M\,r + D\,r\cos\theta}$$

$$r\,(1 + D/(G\,M)\cos\theta) = L^2\big/(G\,M)$$

This is the equation of an ellipse. Identify

$$e = D/(G\,M)$$

$$\boxed{L^2 = G\,M\,a\left(1 - e^2\right)}$$

Q: Consider an orbit with a low angular momentum. In PHY183, you learned that angular momentum is $L/m = v$ (moment arm). At aphelion (farthest from the sun), the angular momentum is low because

A. the moment arm is small.

B. the velocity is small.

C. both the velocity and moment arm are small.

D. the velocity is big.

Q: Consider an orbit with a low angular momentum. When the planet is halfway between perihelion and aphelion, the angular momentum is low because ... (Same foils.)

Energy at perihelion

$$E = \frac{1}{2} v_p^2 - G M \Big/ r_p$$

$$r_p E = \frac{1}{2} L^2 \Big/ r_p - G M = \frac{1}{2} \frac{L^2}{a (1-e)} - G M$$

Add to energy at aphelion

$$\left(r_p + r_a\right) E = \frac{1}{2} \frac{L^2}{a} \frac{2}{1-e^2} - 2 G M$$

$$E = \frac{1}{2} \frac{L^2}{a^2 \left(1-e^2\right)} - G M / a$$

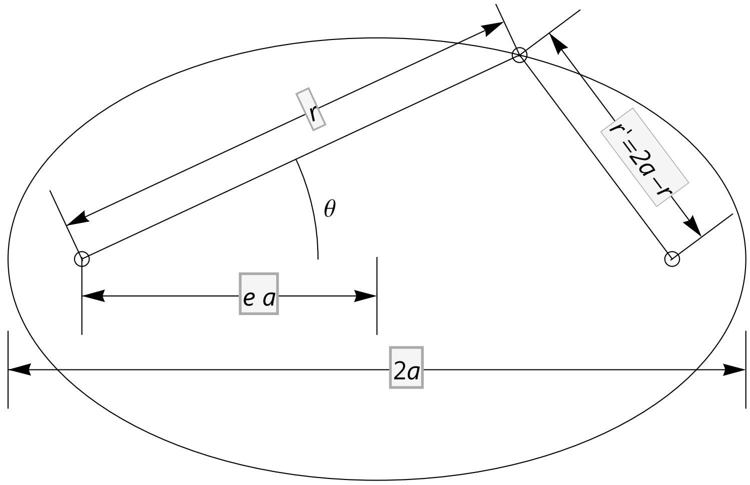
Substitute to get

$$E = \frac{1}{2} G M / a - G M / a$$

$$\boxed{E = -\frac{1}{2} G M / a}$$

Q: State this in the simplest possible way, so that you will remember it.

Definitions used in celestial mechanics



Pericenter location on the orbit that is closest to the big mass.
Apocenter location ... farthest...

What are the perihelion, aphelion, perijove, and periselenium?

True longitude θ referenced to a fixed direction in space.
Longitude of pericenter ϖ (curly pi)
True anomaly $f = \theta - \varpi$ referenced to the pericenter.

Mercury’s orbit precesses 43arcsec/century because of Einstein’s modifications of Newton’s mechanics. The longitude of pericenter changes.

Q: For the figure above, the true longitude is

A. 30°
B. 210°
C. impossible to determine.

Q: For the figure above, the true anomaly is