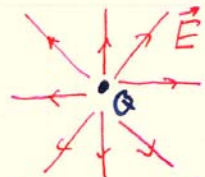
MAXWELL'S EQUATIONS

(1) $\nabla \cdot \vec{E} = \rho/\epsilon_0$

$\epsilon_0 = 8.85 \times 10^{-11} \frac{C}{Vm}$

$\oint_S \vec{E} \cdot d\vec{A} = Q/\epsilon_0$

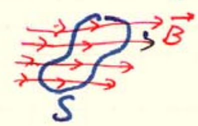


$\vec{E}(\vec{x}) = \frac{Q\hat{r}}{4\pi\epsilon_0 r^2}$

(2) $\nabla \cdot \vec{B} = 0$

$\oint_S \vec{B} \cdot d\vec{A} = 0$

There are no magnetic monoples.

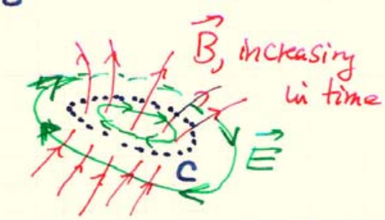


i.e., closed surface S

(3) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{A}$

i.e., open surface S bounded by C



Electromagnetic induction

(4) $\nabla \times \vec{B} = \mu_0 \vec{J}$

(Ampere's Law)

$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$

But Maxwell recognized that these equations are <u>inconsistent</u>:

$\nabla \cdot (\nabla \times \vec{f}) = 0$ for any function $\vec{f}$.

Then eq. (4) implies $\nabla \cdot \vec{J} = 0$.

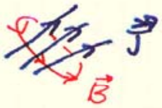
But in general, $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \neq 0$

The Displacement Current

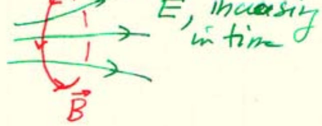
So Maxwell made a leap of faith ...

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad (4)$$

Here $\vec{J}$ = charge current



and $\epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$ = displacement current



The theory becomes self-consistent

$$\nabla \cdot (\nabla \times \vec{B}) = 0$$

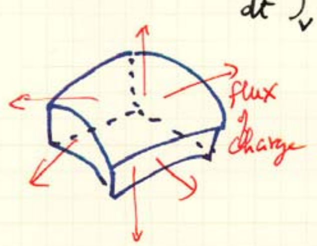
requires $\quad \nabla \cdot \vec{J} + \nabla \cdot \left(\epsilon_0 \dfrac{\partial \vec{E}}{\partial t}\right) = 0.$

$$2^{\text{nd}} \text{ term} = \epsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \vec{E}) = \epsilon_0 \frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0}\right) = \frac{\partial \rho}{\partial t}$$

i.e., $\quad \nabla \cdot \vec{J} + \dfrac{\partial \rho}{\partial t} = 0.$

But that is true. It's the continuity equation, expressing conservation of charge

$$\oint_S \vec{J} \cdot d\vec{A} = -\frac{d}{dt}\int_V \rho \, d^3x = -\frac{dQ}{dt}$$



$$\frac{dQ}{dt} = -\oint \vec{J} \cdot d\vec{A}$$

Q.E.D.

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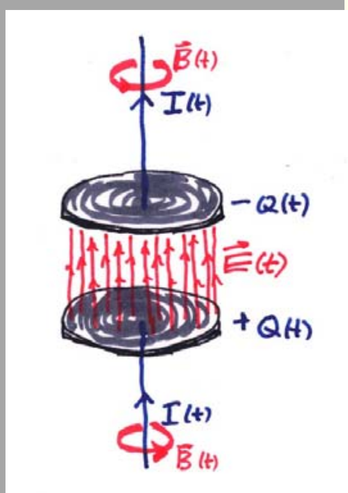
Maxwell's theory, published ~1864, was not widely accepted until the experiments of Heinrich Hertz, done ~1882; because the displacement current was not proven in experiments.



A consequence of the displacement current

Consider a capacitor with a time-dependent current.

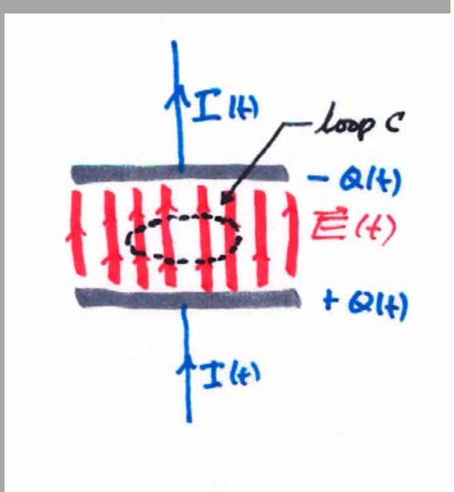
Between the plates,

$\vec{D} = \sigma \hat{k}$ where $\sigma = \dfrac{Q(t)}{\pi a^2}$

$\vec{E} = \dfrac{\sigma}{\epsilon_0} \hat{k}$

/ using the quasi-static approximation; i.e. $Q(t)$ changes slowly /

Determine $\vec{B}(\vec{x}, t)$ between the plates



$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 I + \mu_0 \underbrace{\epsilon_0 \frac{d}{dt}\int_S \vec{E}\cdot d\vec{A}}_{\text{displacement current}}$$

$B_\phi^{(r)} \cdot 2\pi r$; charge current $= 0$

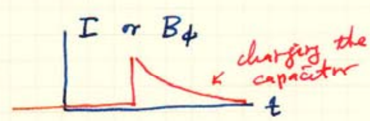
$= \epsilon_0 \dfrac{\dot{\sigma}}{\epsilon_0} \begin{cases} \pi r^2 \\ \text{or} \\ \pi a^2 \end{cases}$

and $\dot{\sigma} = \dfrac{\dot{Q}}{\pi a^2} = \dfrac{I}{\pi a^2}$

Result

$$B_\phi(r,t) = \begin{cases} \dfrac{\mu_0 I r}{2\pi a^2} & \text{for } r \leq a \\ \dfrac{\mu_0 I}{2\pi r} & \text{for } r \geq a \end{cases}$$

but it's hard to measure



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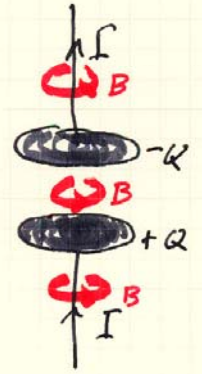


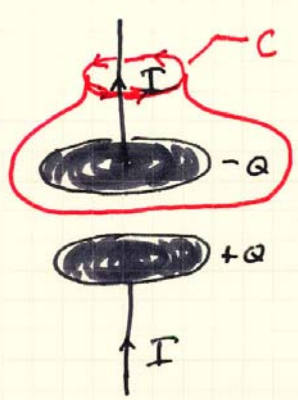
$\leftarrow B_\phi = \dfrac{\mu_0 I}{2\pi r}$

$$B_\phi = \frac{\mu_0 I}{2\pi} \times \begin{cases} r/a^2 & \text{for } r \leq a, \text{ or} \\ 1/r & \text{for } r \geq a \end{cases}$$

$\leftarrow B_\phi = \dfrac{\mu_0 I}{2\pi r}$

(in the quasi-static approximation)

Again, the displacement current means that the time-dependent equations are self-consistent.

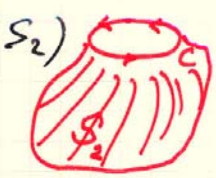


Ampère's law

$$\oint_C \vec{B} \cdot d\vec{l}$$

$$= \mu_0 I(\text{through } S_1)$$

$$= \mu_0 I(\text{through } S_2)$$



is true because

I = charge current

+ displacement current

Maxwell's Equations [1, 2]

$$\nabla \cdot \vec{E} = \rho/\epsilon_0 \quad \text{and} \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{and} \quad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

[1] for isolated charges and currents in freespace.

[2] written in the vector form developed later by Oliver Heaviside.

⇒ 8 coupled equations for the fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$; self consistent because

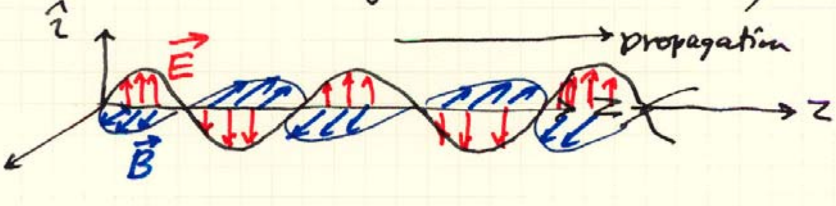
$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \text{/the continuity equation for charge/}$$

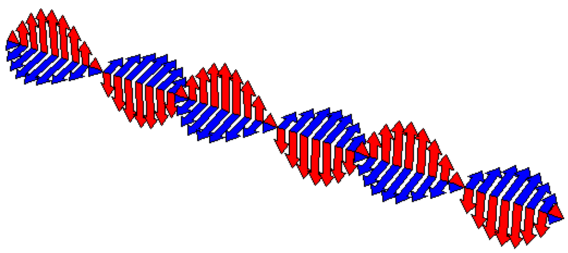
Electromagnetic Waves in empty Space

($\rho = 0$ and $\vec{J} = 0$)

Example $\vec{E}(\vec{x},t) = \hat{i}\, E_0 \sin(kz - \omega t)$

$\vec{B}(\vec{x},t) = \hat{j}\, B_0 \sin(kz - \omega t)$





Polarized in the x direction, propagating in the z direction: $\quad$ 11.1/6

$$\vec{E}(\vec{x},t) = \hat{\imath}\, E_0 \sin(kz-\omega t)$$
$$\vec{B}(\vec{x},t) = \hat{\jmath}\, B_0 \sin(kz-\omega t)$$

- $\nabla\cdot\vec{E} = 0 \qquad \nabla\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$ ✓
- $\nabla\cdot\vec{B} = 0 \qquad$ similarly ✓
- $\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \partial/\partial z \\ E_x & 0 & 0 \end{vmatrix} = \hat{\jmath}\,\frac{\partial E_x}{\partial z} = \hat{\jmath}\, kE_0 \cos(kz-\omega t)$

$$= -\frac{\partial \vec{B}}{\partial t} = +\hat{\jmath}\,\omega B_0 \cos(kz-\omega t)$$

requires $\quad kE_0 = \omega B_0$

- $\nabla\times\vec{B} = \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$

similarly, requires $\quad kB_0 = \mu_0\epsilon_0\,\omega E_0$

Properties of the electromagnetic wave solution.
We have

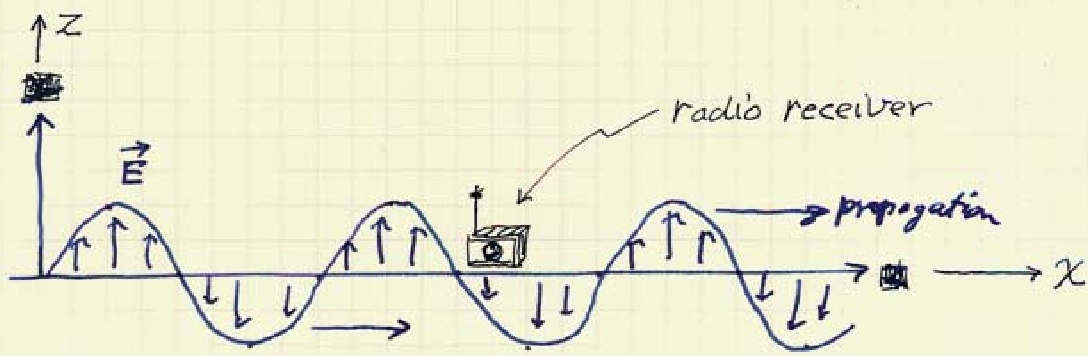
$$\frac{B_0}{E_0} = \frac{k}{\omega} = \frac{\mu_0\epsilon_0\,\omega}{k}.$$

$$\therefore \quad \frac{\omega^2}{k^2} = \frac{1}{\mu_0\epsilon_0} \quad \text{or} \quad \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

$$\boxed{\frac{\omega}{k} = \text{The phase velocity} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c}$$

And $\boxed{B_0 = \frac{E_0}{c}}$

Quiz



A radio wave, polarized in the z direction and traveling in the x direction, has

$$\vec{E}(\vec{x},t) = E_0 \hat{k} \sin(kx-\omega t)$$

$$\vec{B}(\vec{x},t) = -B_0 \hat{j} \sin(kx-\omega t).$$

(A) Determine the energy flux (= Poynting vector, averaged over one period of oscillation) expressed in terms of E_0.

S = {formula}

(B) Now suppose the energy flux is $\frac{0.01}{4\pi} \frac{W}{m^2}$. Calculate E_0.

E_0 = {value} = {number w/ unit}