The Electromagnetic Field Tensor

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

Table 12.3: Lorentz transformations of various quantities. The inertial frame $\mathcal{F}'$ moves with velocity $\mathbf{v}$ with respect to frame $\mathcal{F}$. The components denoted $\|$ and $\perp$ are parallel and perpendicular to $\mathbf{v}$.

coordinates	
$t' = \gamma(t - vx_\| /c^2)$	$t = \gamma(t' + vx'_\| /c^2)$
$x'_\| = \gamma(x_\| - vt)$	$x_\| = \gamma(x'_\| + vt')$
$\mathbf{x}'_\perp = \mathbf{x}_\perp$	$\mathbf{x}_\perp = \mathbf{x}'_\perp$
energy and momentum	
$E' = \gamma(E - vp_\|)$	$E = \gamma(E' + vp'_\|)$
$p'_\| = \gamma(p_\| - vE/c^2)$	$p_\| = \gamma(p'_\| + vE'/c^2)$
$\mathbf{p}'_\perp = \mathbf{p}_\perp$	$\mathbf{p}_\perp = \mathbf{p}'_\perp$
velocity	
$u'_\| = (u_\| - v)/(1 - vu_\| /c^2)$	$u_\| = (u'_\| + v)/(1 + vu'_\| /c^2)$
$\mathbf{u}'_\perp = (1/\gamma)\mathbf{u}_\perp/(1 - vu_\| /c^2)$	$\mathbf{u}_\perp = (1/\gamma)\mathbf{u}'_\perp/(1 + vu'_\| /c^2)$
electric and magnetic fields	
$E'_\| = E_\|$	$E_\| = E'_\|$
$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}_\perp)$	$\mathbf{E}_\perp = \gamma(\mathbf{E}'_\perp - \mathbf{v} \times \mathbf{B}'_\perp)$
$B'_\| = B_\|$	$B_\| = B'_\|$
$\mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \mathbf{v} \times \mathbf{E}_\perp/c^2)$	$\mathbf{B}_\perp = \gamma(\mathbf{B}'_\perp + \mathbf{v} \times \mathbf{E}'_\perp/c^2)$

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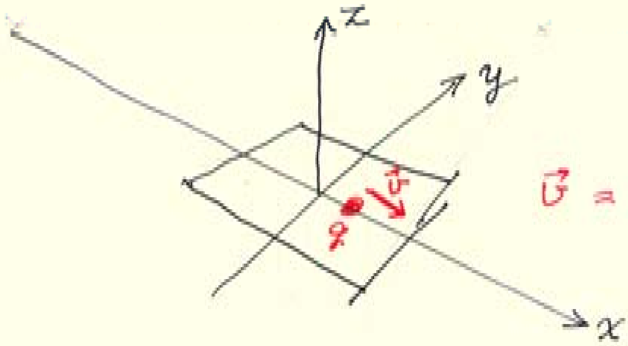
<u>The Fields of a Moving Charge</u>

A very basic question : What are the fields, $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$, due to a charged <u>particle</u> that moves with constant velocity ?

small object



$\vec{v} = v\hat{i}$

Suppose the charge q moves with velocity $\vec{v} = v\hat{i}$ on the x-axis of the inertial frame $\mathcal{F}$.

The position of q is $\vec{x}_q = vt\,\hat{i}$.

The problem is to determine $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$.

We'll find the answer by applying the Lorentz transformation.

- Let $\mathcal{F}'$ be the <u>rest frame</u> of q, in which q is at rest at the origin. Then $\vec{x}_q' = 0$.

In this frame of reference the fields are

$$\vec{E}'(\vec{x}') = \frac{q}{4\pi\epsilon_0}\,\frac{\vec{x}'}{|\vec{x}'|^3} \quad \text{and} \quad \vec{B}'(\vec{x}') = 0.$$

(Coulomb field)

Lorentz transformation of coordinates 12.6/2

$$x^\mu = \Lambda'^{\mu}{}_{\nu}\, x'^\nu \quad \text{where} \quad \Lambda'^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- $x^0 = \Lambda'^{0}{}_{\nu} x'^\nu = \Lambda'^{0}{}_{0} x'^0 + \Lambda'^{0}{}_{1} x'^1$

$$= \gamma x'^0 + \beta\gamma x'^1$$

$$t = \gamma\left(t' + \frac{v}{c^2} x'\right)$$

$(x'=0)$

The time coordinate of the particle is $t = \gamma t'$;
time dilation : $\delta t = \gamma\, \delta t' > \delta t'$.

- $x^1 = \Lambda'^{1}{}_{\nu} x'^\nu = \Lambda'^{1}{}_{0} x'^0 + \Lambda'^{1}{}_{1} x'^1$

$$= \beta\gamma c t' + \gamma x'$$

$$x = \gamma(x' + vt')$$

$(x'=0)$

The x coordinate of the particle is
$x = \gamma v t' = vt$; as specified, the particle moves on the x axis with velocity $v\hat{x}$.

- $y = y'$ and $z = z'$

The particle has $\vec{x} = (vt, 0, 0)$.

- Note that $|\vec{x}'| = \sqrt{(x')^2 + (y')^2 + (z')^2}$

$$= \sqrt{\gamma^2 (x - vt)^2 + y^2 + z^2}$$

Check : $\gamma(x - vt) = \gamma \cdot \gamma(x' + vt') - \gamma v \gamma\left(t' + \frac{v}{c^2} x'\right)$

$$= \gamma^2(1 - \beta^2) x' = x' \quad ✓$$

Lorentz transformation of the fields

12.6/3

$$F^{\mu\nu} = \Lambda'^{\mu}{}_{\alpha} \Lambda'^{\nu}{}_{\beta} F'^{\alpha\beta}$$

The Electric field

$$E^i = c F^{0i} = c \Lambda'^{0}{}_{\alpha} \Lambda'^{i}{}_{\beta} F'^{\alpha\beta} \ \ldots$$

I could go on, but I already did this last time.

x-component $\quad E_{\parallel} = E'_{\parallel}$

$$E_x = E'_x = \frac{q}{4\pi\epsilon_0} \frac{x'}{|\vec{x}'|^3} = \frac{q}{4\pi\epsilon_0} \frac{\gamma(x - vt)}{D^3}$$

$$\text{let } D = \sqrt{\gamma^2 (x-vt)^2 + y^2 + z^2} \leftarrow |\vec{x}'|$$

y and z component $\quad \vec{E}_{\perp} = \gamma(\vec{E}'_{\perp} - \vec{v} \times \vec{B}'_{\perp}) = \gamma \vec{E}'_{\perp}$

0

$$E_y = \gamma \frac{q}{4\pi\epsilon_0} \frac{y'}{|\vec{x}'|^3} = \frac{\gamma q y}{4\pi\epsilon_0 D^3}$$

$$E_z = \gamma \frac{q}{4\pi\epsilon_0} \frac{z'}{|\vec{x}'|^3} = \frac{\gamma q z}{4\pi\epsilon_0 D^3}$$

$$\vec{E}(\vec{x}, t) = \frac{q\gamma}{4\pi\epsilon_0} \frac{(x - vt)\hat{i} + y\hat{j} + z\hat{k}}{D^3}$$

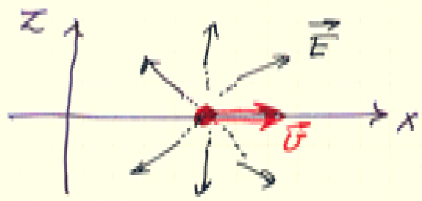
$$\vec{E}(\vec{x},t) = \frac{q\gamma}{4\pi\epsilon_0}\,\frac{(x-vt)\hat{i}+y\hat{j}+z\hat{k}}{D^3} \qquad 12.6/4$$

$$D = \sqrt{\gamma^2(x-vt)^2+y^2+z^2}$$

Comments

(1) If $\vec{v}=0$ the $\vec{E}$ is the Coulomb field.

(2) The direction of $\vec{E}(\vec{x},t)$ is radially away from the simultaneous position of q /no time delay in the field—/



(3) The magnitude of $\vec{E}$ for points on the x axis is

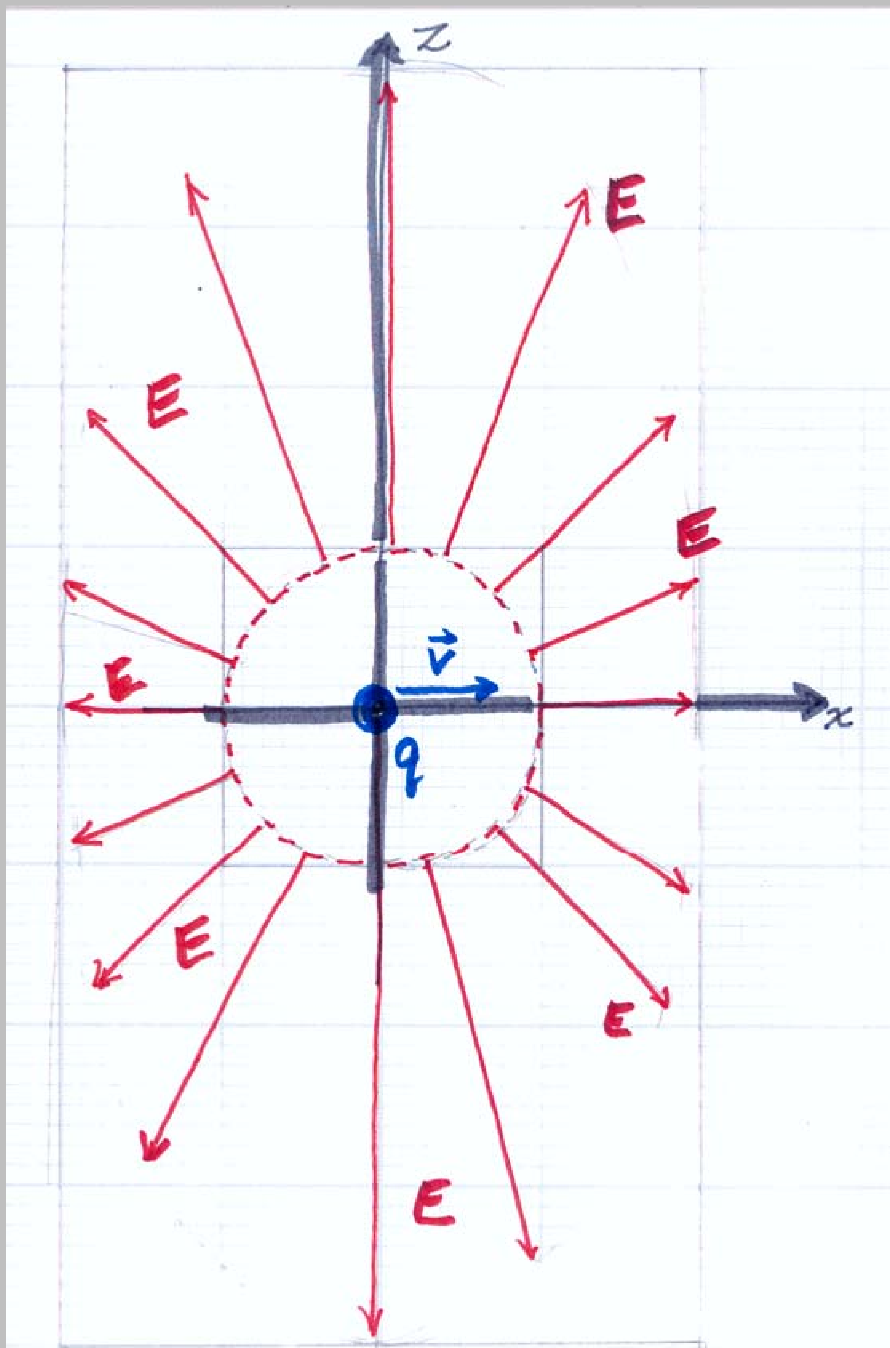
$$E(x,0,0) = \frac{q\gamma}{4\pi\epsilon_0}\,\frac{x-vt}{\gamma^3(x-vt)^3} = \frac{1}{\gamma^2}\,\frac{q}{4\pi\epsilon_0 d^2}$$

where d = the distance from x to q, i.e., $d = x-vt$.
the magnitude is less than the Coulomb field by the factor $1/\gamma^2$. (points on x axis)

(4) the magnitude of $\vec{E}$ for points on the plane $x=x_q=vt$

$$E(vt,y,z) = \frac{q\gamma}{4\pi\epsilon_0}\,\frac{\sqrt{y^2+z^2}}{(y^2+z^2)^{3/2}} = \gamma\,\frac{q}{4\pi\epsilon_0 d^2}$$

where d = the distance to the charge q, i.e., $d=\sqrt{y^2+z^2}$.
This magnitude is greater than the Coulomb field by the factor γ. (points on the plane $x=x_q$)


z
E
E
E
E
$\vec{v}$
x
q
E
E
E

12/6/5

The Magnetic Field

$$B_x = F^{23} = \Lambda^{2}{}_{\alpha} \Lambda^{3}{}_{\beta} F'^{\alpha\beta} = F'^{23} = B'_x \cdots$$

I could continue, but I already did this last time.

x-component $\quad B_x = B'_x = 0$

y and z components $\quad \vec{B}_\perp = \gamma(\underbrace{\vec{B}'_\perp}_{0} + \frac{\vec{v}}{c^2} \times \underbrace{\vec{E}'_\perp}_{\text{Coulomb field}})$

$$\vec{B}_\perp = \gamma \frac{v}{c^2} \hat{i} \times (E'_y \hat{j} + E'_z \hat{k})$$

$$= \frac{\gamma v}{c^2} (E'_y \hat{k} - E'_z \hat{j})$$

$$= \frac{\gamma v}{c^2} \frac{q}{4\pi\epsilon_0} \frac{y' \hat{k} - z' \hat{j}}{D^3}$$

$$\vec{E}' = \frac{q}{4\pi\epsilon_0} \frac{\vec{x}'}{|\vec{x}'|^3}$$

$$\vec{B}_\perp = \frac{\mu_0}{4\pi} \frac{\gamma v q}{D^3} (-z \hat{j} + y \hat{k})$$

$$D = \sqrt{\gamma^2 (x - vt)^2 + y^2 + z^2}$$

Currents

(1) The direction of $\vec{B}$ is azimuthally around the x-axis

$\vec{B}$ curls around the x axis

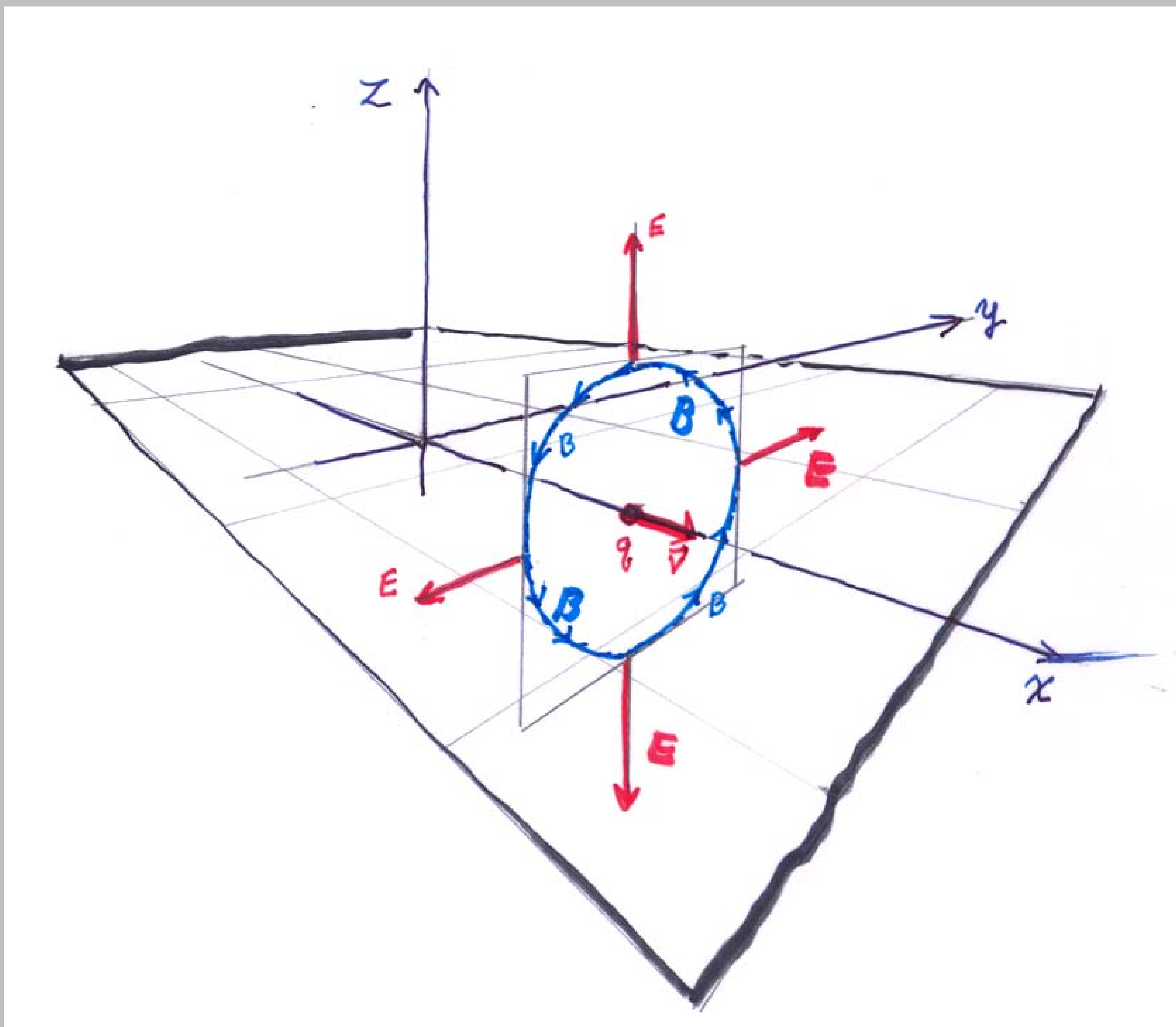
(2) On the x-axis, $\vec{B} = 0$.

(3) On the plane $x = x_q$ $(= vt)$, $\vec{B} = \frac{\mu_0}{4\pi} \frac{\gamma v q}{D^2} \hat{\psi}$

(4) On the plane $x = x_q$,

$$\frac{B}{E} = \frac{\frac{\mu_0}{4\pi} \frac{\gamma v q}{d^2}}{\gamma \frac{q}{4\pi\epsilon_0 d^2}} = \frac{v}{c^2}$$

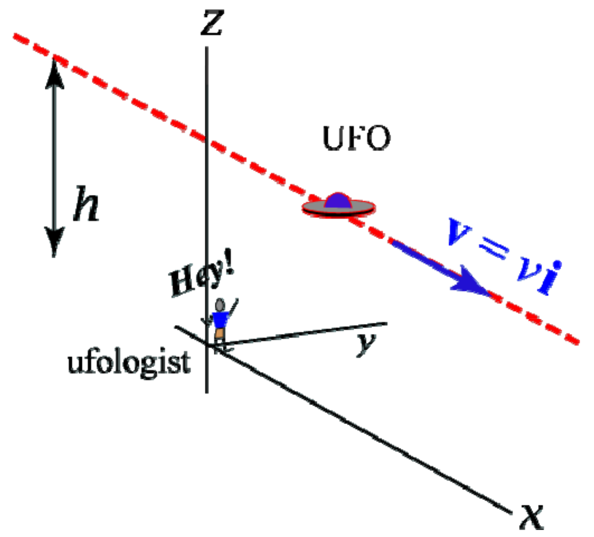
(Recall: for an e.m. wave, $B/E = 1/c$.)


Z
E
y
B
B
E
E
q
$\vec{v}$
B
B
x
E

Quiz Question

A UFO flies over head. It has electric charge Q = 10,000 coulombs. It moves with speed 0.9c parallel to the x axis, at height 1,000 m above the ground. The ufologist on the ground is ready for it, and measures the electric and magnetic fields. Calculate the field measurements when the UFO is directly above the ufologist.

(A) Electric field (B) Magnetic field



The UFO has electrical charge 10,000 C. It is moving with a speed of 0.9 c. Calculate the fields observed by the ufologist.