Magnetism from Relativity

Start with 2 basic "axioms"

(1) A point charge at rest has $\vec{E} = \dfrac{q\hat{r}}{4\pi\epsilon_0 r^2}$ Coulomb field, measured by M. Coulomb in the 18th cent.

(2) The Principles of Special Relativity

The full Maxwell theory of electromagnetism follows from these 2 "axioms". So, in a sense, magnetism is a consequence of relativity. We'll need....

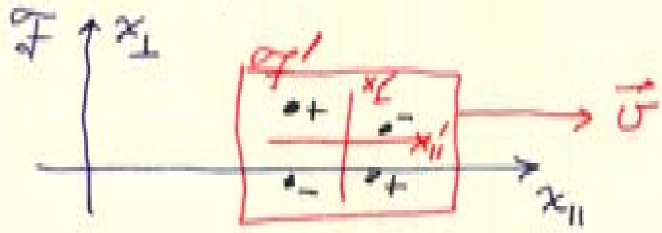
- Lorentz frame $\mathcal{F}'$ of an electrostatic system; i.e., charges at rest and $\vec{E}'(\vec{x}')$ in frame $\mathcal{F}'$.

- A small test charge q;

$$\frac{d\vec{p}\,'}{dt'} = \vec{F}' = q\,\vec{E}'(\vec{x}')$$

no magnetism in frame $\mathcal{F}'$

- Another arbitrary reference frame $\mathcal{F}$; say $\mathcal{F}'$ moves with velocity $\vec{v}$ relative to $\mathcal{F}$.



We'll determine the equations of motion for the test charge q in the coordinates of frame $\mathcal{F}$, by Lorentz transformations, $\mathcal{F}' \to \mathcal{F}$

Derive the equation of motion in frame $\mathcal{F}$, based on relativity.

$$\frac{d\vec{p}}{dt} = \vec{F} = \begin{cases} F_{\parallel} & \text{in direction} \parallel \text{to } \vec{v} \\ \vec{F}_{\perp} & \text{in directions} \perp \text{to } \vec{v} \end{cases}$$

Lorentz T.

$$x_{\parallel} = \gamma(x_{\parallel}' + vt')$$
$$\vec{x}_{\perp} = \vec{x}_{\perp}'$$
$$t = \gamma\left(t' + \frac{v}{c^2} x_{\parallel}'\right)$$
$$\mathcal{F} \leftarrow \mathcal{F}'$$

& same for any 4-vector

- PARALLEL COMPONENT

$$F_{\parallel} = \frac{dp_{\parallel}}{dt} = \frac{\gamma\left(dp_{\parallel}' + \frac{v}{c^2} dE'\right)}{\gamma\left(dt' + \frac{v}{c^2} dx_{\parallel}'\right)}$$

because

$$x^{\mu} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \text{ and } p^{\mu} = \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix} \text{ are 4-vectors.}$$

The work–kinetic energy theorem says

$$dE' = \vec{F}' \cdot d\vec{x}' \quad \text{(displacement of } q\text{)}$$

← the test charge

$$dE' = \vec{F}' \cdot \vec{u}'\, dt'$$

where $\vec{u}'$ (and $\vec{u}$ in frame $\mathcal{F}$) is the particle velocity of q.

N.B. $\vec{u}'$ and $\vec{u}$ are not $\vec{v}$!! (why?)

$$F_{\parallel} = \frac{F_{\parallel}' + \frac{v}{c^2} \vec{F}' \cdot \vec{u}'}{1 + \frac{v}{c^2} u_{\parallel}'}$$

← also substitute $\vec{F}' \cdot \vec{u}' = F_{\parallel}' u_{\parallel}' + \vec{F}_{\perp}' \cdot \vec{u}_{\perp}'$

The addition of velocities (Table 12.3)

$$u_{\parallel} = \frac{v + u_{\parallel}'}{1 + \frac{v u_{\parallel}'}{c^2}} \quad \text{and} \quad \vec{u}_{\perp} = \frac{\vec{u}_{\perp}'}{\gamma\left(1 + \frac{v u_{\parallel}'}{c^2}\right)}$$

$$F_{\parallel} = F_{\parallel}' + \frac{v\gamma}{c^2} \vec{F}_{\perp}' \cdot \vec{u}_{\perp} \qquad (1)$$

- PERPENDICULAR COMPONENT

$$\vec{F}_\perp = \frac{d\vec{p}_\perp}{dt} = \frac{d\vec{p}_\perp{}'}{\gamma(dt' + \frac{v}{c^2}dx'_{\parallel})}$$

$$\vec{F}_\perp = \frac{1}{\gamma}\frac{\vec{F}_\perp{}'}{1+\frac{vu'_{\parallel}}{c^2}} \qquad (2)$$

Now, we DEFINE the electric and magnetic fields in the Lorentz frame $\mathcal{F}$ by their forces on q —

$$\vec{F} = q\vec{E} + q\,\vec{u}\times\vec{B} \qquad (3)$$

$\vec{u}$ is the velocity of the test charge

Starting from (1) and (2), ← (results of Relativity)

with $\vec{F}' = q\vec{E}'$, ← (Coulomb's law of electrostatics)

our task is to construct $\vec{E}$ and $\vec{B}$ defined in (3).

The parallel component of Eq. (3)

$\longrightarrow$ Consider $\vec{v}\cdot\vec{F} = vF_{\parallel}$

③ By Eq. (3)

$$\vec{v}\cdot\vec{F} = vF_{\parallel} = q v E_{\parallel} + q \underbrace{\vec{v}\cdot(\vec{u}\times\vec{B})}$$

$$= \epsilon_{ijk}\, v_i \cdot u_j \cdot B_k = \vec{u}\cdot(\vec{B}\times\vec{v})$$

$$= \vec{u}\cdot(\vec{B}_\perp\times\vec{v}) = \vec{u}_\perp\cdot(\vec{B}_\perp\times\vec{v})$$

$$vF_{\parallel} = q v E_{\parallel} + q\,\vec{u}_\perp\cdot(\vec{B}_\perp\times\vec{v})$$

④ Compare Eq. (1)

$$F_{\parallel} = qE_{\parallel}' + \frac{v\gamma}{c^2}\, q\,\vec{E}_\perp'\cdot\vec{u}_\perp .$$

The two equations must agree for <u>any $\vec{v}$</u> and <u>any $\vec{u}$</u>.

The only way is to have

$$E_{\parallel} = E_{\parallel}' \qquad \text{and} \qquad \underbrace{\vec{B}_\perp\times\vec{v} = \frac{v^2}{c^2}\gamma\,\vec{E}_\perp'}_{\text{or, equivalently}}$$

$$\vec{v}\times(\vec{B}_\perp\times\vec{v}) = \vec{B}_\perp v^2 = \frac{v^2}{c^2}\gamma\,\vec{v}\times\vec{E}_\perp'$$

$$\vec{B}_\perp = \frac{\gamma}{c^2}\,\vec{v}\times\vec{E}_\perp'$$

RESULTS

$$\boxed{E_{\parallel} = E_{\parallel}' \quad \text{and} \quad \vec{B}_\perp = \frac{\gamma}{c^2}\,\vec{v}\times\vec{E}_\perp'}$$

The perpendicular components of $\vec{E}_\gamma$ (3)

$\hookrightarrow$ consider $\vec{v} \times \vec{F} = \vec{v} \times \vec{F}_\perp$

Identity $$1 - \frac{v u_\parallel}{c^2} = \frac{1}{\gamma^2} \frac{1}{1 + \frac{v u'_\parallel}{c^2}}$$

Proof $$\text{L.H.S.} = 1 - \frac{v}{c^2} \frac{v + u'_\parallel}{1 + v u'_\parallel / c^2} \quad \text{(addition of velocities)}$$

$$= \left\{ 1 + \frac{v u'_\parallel}{c^2} - \frac{v}{c^2}(v + u'_\parallel) \right\} \frac{1}{1 + v u'_\parallel / c^2}$$

$$= \left(1 - \frac{v^2}{c^2}\right) \frac{1}{1 + v u'_\parallel / c^2} = \text{R.H.S.} \quad \underline{\text{QED}}$$

① By Eq. (3) $\vec{F} = q\vec{E} + q\,\vec{u} \times \vec{B}$

$$v \times \vec{F}_\perp = q\,\vec{v} \times \vec{E}_\perp + q\,\underbrace{\vec{v} \times (\vec{u} \times \vec{B})}$$

$$= \vec{u}\;\vec{v}\cdot\vec{B} - \vec{B}\;\vec{v}\cdot\vec{u}$$

$$= \vec{u}\;v B_\parallel - \vec{B}\;v u_\parallel$$

$$= \vec{u}_\perp v B_\parallel + u_\parallel \vec{v} B_\parallel - B_\parallel \vec{v} u_\parallel - \vec{B}_\perp v u_\parallel$$

cancel

$$= \vec{u}_\perp v B_\parallel - \vec{B}_\perp v u_\parallel$$

$$\vec{v} \times \vec{F}_\perp = q\,\vec{v} \times \vec{E}_\perp + q\,v B_\parallel \vec{u}_\perp - q\,v u_\parallel \underbrace{\frac{\gamma}{c^2} \vec{v} \times \vec{E}'_\perp}_{\vec{B}_\perp \text{ already known}}$$

② Compare to Eq. (2)

$$\vec{v} \times \vec{F}_\perp = \frac{q\,\vec{v} \times \vec{E}'_\perp}{\gamma\left(1 + \frac{v u'_\parallel}{c^2}\right)} = q\,\vec{v} \times \vec{E}'_\perp\;\gamma\left(1 - \frac{v u_\parallel}{c^2}\right)$$

The equations must hold for all $\vec{v}$ and all $\vec{u}$; that requires:

$$B_\parallel = 0 \quad \text{and} \quad q\,\vec{v} \times \vec{E}_\perp = q\,\vec{v} \times \vec{E}'_\perp \left\{ \gamma\left(1 - \frac{v u_\parallel}{c^2}\right) + \frac{\gamma}{c^2} v u_\parallel \right\}$$

i.e., $\vec{E}_\perp = \gamma \vec{E}'_\perp$

RESULTS :

$$\boxed{B_\parallel = 0 \quad \text{and} \quad \vec{E}_\perp = \gamma \vec{E}'_\perp}$$

RESULT

test charge with velocity $\vec{u}$

The force on q, with respect to the coordinates of frame $\mathcal{F}$, is

$$\vec{F} = q\vec{E} + q\,\vec{u} \times \vec{B}$$

(Note the magnetic force!)

where

$$E_{\parallel} = E_{\parallel}' \qquad B_{\parallel} = 0$$

AND

$$\vec{E}_{\perp} = \gamma \vec{E}_{\perp}' \qquad \vec{B}_{\perp} = \frac{\gamma}{c^2}\vec{v} \times \vec{E}_{\perp}'$$

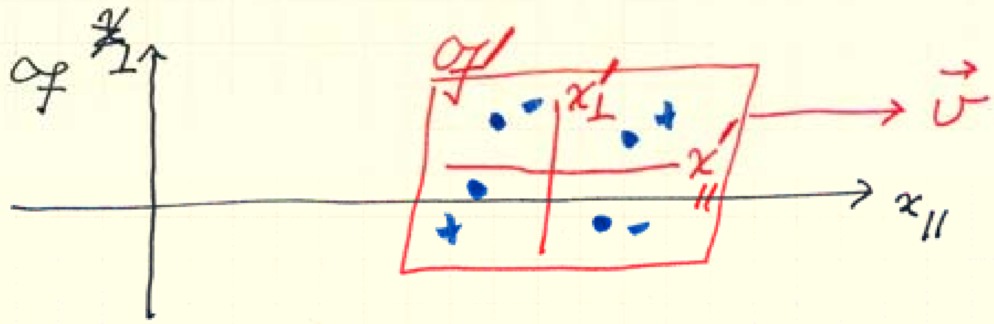
This is just the field transformation for the special case of an electrostatic system, $\vec{E}'(\vec{x}')$ and $\vec{B}' = 0$

Magnetism (the force $q\vec{u}\times\vec{B}$) is a necessary consequence of the principles of special relativity.

Magnetism from Relativity

Quiz

12.7/7



Suppose $\nabla' \cdot \vec{E}' = \dfrac{\rho'(\vec{x}')}{\epsilon_0}$ and $\nabla' \times \vec{E}' = 0$;

i.e., we have an electrostatic system relative to the inertial frame $\mathcal{F}'$.

We showed that relative to frame $\mathcal{F}$

$$\vec{B}(\vec{x}, t) = \frac{\gamma}{c^2}\, \vec{v} \times \vec{E}'_{\perp}(\vec{x}')$$

(A) Calculate $\nabla \times \vec{B}$.

(B) Describe the result in words.