The energy–momentum flux tensor

$$T^{\mu\nu} = \frac{1}{\mu_0}\left\{ F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma}F_{\rho\sigma} \right\}$$

$\sum_{\rho=0}^{3}$ implied $\qquad \sum_{\rho}\sum_{\sigma}$ implied

$$F^{\mu\rho}F^{\nu}{}_{\rho} = F^{\mu\rho}F^{\nu\sigma}g_{\rho\sigma} \quad \leftarrow \sum_{\rho}\sum_{\sigma}$$

This is a tensor with indices $\mu\nu$. (ρ, σ are summed)

$$F^{\mu\nu} = \begin{array}{c} \nu \rightarrow \\ \mu \downarrow \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix} \end{array} \qquad g^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathrm{diag}(-1, 1, 1, 1)$$

Symmetry properties

- $g^{\mu\nu}$ is symmetric : $g^{\nu\mu} = g^{\mu\nu}$
- $F^{\mu\nu}$ is antisymmetric : $F^{\nu\mu} = -F^{\mu\nu}$
- $T^{\mu\nu}$ is symmetric :

$$T^{\nu\mu} = \frac{1}{\mu_0}\left\{ \underbrace{F^{\nu\rho}F^{\mu\sigma}g_{\rho\sigma}} - \frac{1}{2} \underbrace{g^{\nu\mu}}_{g^{\mu\nu}} F^{\rho\sigma}F_{\rho\sigma} \right\}$$

$$= F^{\mu\sigma}F^{\nu\rho}g_{\rho\sigma}$$

$$= F^{\mu\sigma}F^{\nu\rho}g_{\sigma\rho} = F^{\mu\rho}F^{\nu\sigma}g_{\rho\sigma}$$

(dummy summation variable)

$$= T^{\mu\nu} \quad \text{.....symmetric}$$

The 4-dimensional divergence of $T^{\mu\nu}$ is something interesting

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Recall the field equations in covariant form

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \quad (1)$$

where J^μ = 4 vector current density $(c\rho, J_x, J_y, J_z)$

$$\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \quad (2)$$

where $G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$ (dual tensor)

(1) $\begin{cases} \text{Gauss's law } (\mu=0) \\ \text{Ampère's Law } (\mu=123) \end{cases}$

(2) $\begin{cases} \text{Gauss's Law } \nabla\cdot\vec{B}=0 \ (\mu= \\ \text{Faraday's Law } (\mu=123) \end{cases}$

Now calculate $\dfrac{\partial T^{\mu\nu}}{\partial x^\nu}$

$$\mu_0 T^{\mu\nu} = F^{\mu\rho}F^{\nu\sigma}g_{\rho\sigma} - \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}$$

$$\mu_0 \frac{\partial T^{\mu\nu}}{\partial x^\nu} = \frac{\partial F^{\mu\rho}}{\partial x^\nu}F^{\nu\sigma}g_{\rho\sigma} + F^{\mu\rho}\frac{\partial F^{\nu\sigma}}{\partial x^\nu}g_{\rho\sigma} - \frac{1}{4}g^{\mu\nu}\left[\frac{\partial F^{\rho\sigma}}{\partial x^\nu}F_{\rho\sigma} + F^{\rho\sigma}\frac{\partial F_{\rho\sigma}}{\partial x^\nu}\right]$$

each term is a 4-vector with index μ.

- Second term $= F^{\mu\rho}(-\mu_0 J^\sigma)g_{\rho\sigma} = -\mu_0 F^{\mu\rho}J_\rho$
- Third term = fourth term

$$\boxed{\begin{array}{c} A^\lambda B_\lambda = A_\lambda B^\lambda \\ \text{See why?} \end{array}}$$

$\therefore$ third + fourth

$$= -\frac{1}{2}g^{\mu\nu}F_{\rho\sigma}\frac{\partial F^{\rho\sigma}}{\partial x^\nu}$$

$$= \frac{1}{2}F_{\lambda\kappa}\frac{\partial F^{\kappa\lambda}}{\partial x_\mu}$$

renaming the dummy indices: $\rho \to \lambda$ and $\sigma \to \kappa$

• First term $= \dfrac{\partial F^{\mu\rho}}{\partial x^\nu} F^{\nu\sigma} g_{\rho\sigma}$ 12/9/13

$$= \frac{\partial F^{\mu\rho}}{\partial x_\nu} F_{\nu\rho}$$

Raising and Lowering Indic

$A^\lambda B_\lambda = A_\lambda B^\lambda$

$A^\lambda g_{\lambda\kappa} = A_\kappa$

$A_0 = -A^0 \quad , \quad A_i = A^i$

(a): $= F_{\lambda\kappa} \dfrac{\partial F^{\mu\kappa}}{\partial x_\lambda}$ $\quad \nu \to \lambda$, $\rho \to \kappa$

$$= -F_{\kappa\lambda} \frac{\partial F^{\mu\kappa}}{\partial x_\lambda} = +F_{\kappa\lambda} \frac{\partial F^{\kappa\mu}}{\partial x_\lambda}$$

(b): $= F_{\lambda\kappa} \dfrac{\partial F^{\lambda\mu}}{\partial x_\kappa}$ change the dummy indices: $\kappa \to \lambda$ and $\lambda \to \kappa$

$$= \frac{1}{2} F_{\lambda\kappa} \left\{ \frac{\partial F^{\mu\kappa}}{\partial x_\lambda} + \frac{\partial F^{\lambda\mu}}{\partial x_\kappa} \right\}$$

$a = b$

$\therefore a = \frac{1}{2}(a+b)$

Result

$$\mu_0 \frac{\partial T^{\mu\nu}}{\partial x^\nu} = -\mu_0 F^{\mu\rho} J_\rho$$

(3rd+4th) (1st)

$$+ \frac{1}{2} F_{\lambda\kappa} \left\{ \frac{\partial F^{\kappa\lambda}}{\partial x_\mu} + \frac{\partial F^{\mu\kappa}}{\partial x_\lambda} + \frac{\partial F^{\lambda\mu}}{\partial x_\kappa} \right\}$$

In $\{\ldots\}$ we have 3 cyclic permutations $(\mu\kappa\lambda) + (\lambda\mu\kappa) + (\kappa\lambda\mu)$

But $\epsilon_{\mu\nu\alpha\beta} \dfrac{\partial F^{\alpha\beta}}{\partial x_\nu} = 0$

$$\boxed{\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0}$$

Given μ this is The sum of 3 cyclic permutations of $\nu\alpha\beta$. ($\nu\,\alpha\,\beta$ must all be different)

Exercise: { … } = 0

Result $$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = -F^{\mu\beta} J_\beta$$

This equation is the continuity equation for energy and conservation. IMPORTANT!

Energy and momentum are conserved.
The continuity equation tells how energy and momentum are converted between fields and particles.

The time component ($\mu = 0$)

$$\frac{\partial T^{0\nu}}{\partial x^\nu} = \frac{\partial T^{00}}{\partial x^0} + \frac{\partial T^{0i}}{\partial x^i} = \frac{1}{c}\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x^i}$$

$$= -F^{0\beta} J_\beta = -F^{0i} J_i = -F^{0i} J^i$$

$$= -\frac{E^i}{c} J^i = -\frac{1}{c}\vec{J}\cdot\vec{E}$$

Let $T^{00} = u$ and $T^{0i} = \frac{1}{c} S^i$.

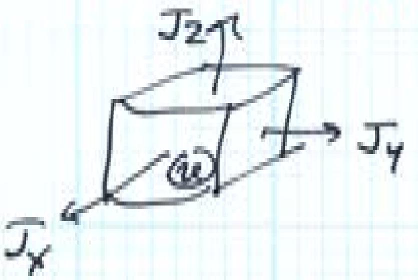
$$\boxed{\frac{\partial u}{\partial t} + \nabla\cdot\vec{S} = -\vec{J}\cdot\vec{E}}$$

$\leftarrow$ $(-1) \times$ work done on charged particles in the current

$\Rightarrow$ energy is locally conserved

$\Rightarrow$ u = energy density $= T^{00}$

$\Rightarrow$ $\vec{S}$ = energy flux $= c\, T^{0i}\, \hat{e}_i$



$$\oint \vec{S}\cdot d\vec{A} = -\frac{d}{dt}\int u\, d^3x - \int \vec{J}\cdot\vec{E}\, d^3x$$

$$\frac{d}{dt}\int u\, d^3x = -\int \vec{S}\cdot d\vec{A} - \int \vec{J}\cdot\vec{E}\, d^3x$$

energy changes $\because$ flows out or does work

Express u and $\vec{S}$ in terms of $\vec{E}$ and $\vec{B}$

$$\mu_0 T^{\mu\nu} = F^{\mu\rho} F^{\nu}{}_{\rho} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$

$\boxed{\mu\nu = 00}$

$$\mu_0 T^{00} = F^{0i} F^{0i} - \frac{1}{4} g^{00} \{ -2F^{0i} F^{0i} + F^{jk} F^{jk} \}$$

$$= \frac{\vec{E}^2}{c^2} - \frac{1}{2} \frac{\vec{E}^2}{c^2} + \frac{1}{2} \vec{B}^2 = \frac{1}{2} \left(\frac{\vec{E}^2}{c^2} + B^2 \right)$$

$$\therefore \quad u = \frac{1}{2} \epsilon_0 \vec{E}^2 + \frac{1}{2\mu_0} \vec{B}^2 \quad (\checkmark)$$

$\boxed{\mu\nu = 0i}$

$$\mu_0 T^{0i} = F^{0j} F^{i}{}_{j} = F^{0j} F^{ij}$$

$$= \frac{E^j}{c} \epsilon_{ijk} B^k = \frac{1}{c} (\vec{E} \times \vec{B})^i$$

$$\therefore \quad \vec{S} = c T^{0i} \hat{e}_i = \frac{\vec{E} \times \vec{B}}{\mu_0} \quad (\checkmark)$$

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Space components ($\mu = i$)

$T^{i0} = T^{0i} = S^i/c$

$$\frac{\partial T^{i\nu}}{\partial x^\nu} = \frac{\partial T^{i0}}{\partial x^0} + \frac{\partial T^{ij}}{\partial x^j} = \frac{1}{c}\frac{\partial T^{i0}}{\partial t} + \frac{\partial T^{ij}}{\partial x^j}$$

$$= -F^{i\rho}J_\rho = -F^{i0}J_0 - F^{ij}J_j$$

$$= -(-E^i)(-c\rho) - F^{ij}J^j$$

$\epsilon_{ijk}B^kJ^j = (\vec{J}\times\vec{B})^i$

$$= -\rho E^i - (\vec{J}\times\vec{B})^i$$

$$\frac{1}{c^2}\frac{\partial S^i}{\partial t} + \frac{\partial T^{ij}}{\partial x^j} = -\rho E^i - (\vec{J}\times\vec{B})^i$$

$$\frac{1}{c^2}\frac{\partial \vec{S}}{\partial t} + \nabla\cdot\overleftrightarrow{T} = -\left[\rho\vec{E} + \vec{J}\times\vec{B}\right]$$

the continuity equation for momentum

$\frac{1}{c^2}\frac{\partial \vec{S}}{\partial t}$ = rate of change of momentum per unit volume

|| $\vec{S}/c^2$ = field momentum density = $\vec{\pi}$

$\nabla\cdot\overleftrightarrow{T}$ = momentum flux per unit volume

|| $\int \nabla\cdot\overleftrightarrow{T}\, d^3x = \oint \overleftrightarrow{T}\cdot d\vec{A}$

momentum flowing away from the point $\vec{x}$

T = the moment per unit area (a matrix)

$\rho\vec{E} + \vec{J}\times\vec{B}$ = force per unit volume on charged particles

$\frac{d\vec{P}}{dt} = \vec{F}$ so $\vec{P}_{\text{particles}} + \int \vec{\pi}\, d^3x$ ~~is constant.~~ = flux

Momentum is conserved and e.m. fields have momentum.

Photons are massless

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	Classical theory	Photon theory
Energy flux	$\vec{S}$	$E_\gamma n_\gamma c$
Momentum density	$\vec{\pi} = \frac{\vec{S}}{c^2}$	$p_\gamma n_\gamma$
Momentum flux	$c\vec{\pi} = \frac{\vec{S}}{c}$	$p_\gamma n_\gamma c$
$\frac{\text{Energy flux}}{\text{Momentum flux}}$	c	$\frac{E_\gamma}{p_\gamma}$

So E_γ / p_γ must be c to be consistent.

$E = \sqrt{p^2c^2 + m^2c^4}$ for a massive particle.

$E = pc$ if the mass is 0.

$\therefore$ Photons are massless.

Quiz Question

z Question 12.9/q

^{ij} = the momentum flux per unit area

^{ij} = the flux of P^i in the direction of x^j.

Express T^{11} in terms of (E_x, E_y, E_z) and $(B_x, B_y,$

Express T^{12} in terms of (E_x, E_y, E_z) and $(B_x, B_y,$