Scalar and Vector Potentials

Ro/1

We've studied Maxwell's equations ...

$$\nabla\cdot\vec{E} = \rho/\epsilon_0 \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = -\partial\vec{B}/\partial t \qquad \nabla\times\vec{B} = \mu_0\left(\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

These describe charges and currents in vacuum; to include material properties of polarization and magnetization, replace $\epsilon_0 \to \epsilon$ and $\mu_0 \to \mu$.

Recall electrostatics $\nabla\cdot\vec{E} = \rho/\epsilon_0$ and $\nabla\times\vec{E} = 0$.

Write $\vec{E} = -\nabla V$, which guarantees $\nabla\times\vec{E} = 0$.

Then $-\nabla^2 V = \rho/\epsilon_0$ (Poisson's equation)

E.g., $V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|} = \int \frac{dQ}{4\pi\epsilon_0 r}$ (3.60)

The field calculation is reduced to integration.

Recall magnetostatics $\nabla\cdot\vec{B} = 0$ and $\nabla\times\vec{B} = \mu_0\vec{J}$.

Write $\vec{B} = \nabla\times\vec{A}$, which guarantees $\nabla\cdot\vec{B} = 0$.

Then $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$.

E.g., $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|} = \int \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}}{r}$ (8.60)

— reduced to integration.

Time dependent fields

R 0/2

$\hookrightarrow \vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$

We'll need both $V(\vec{x},t)$ and $\vec{A}(\vec{x},t)$.
How shall they be defined?
Are they unique? (No! There are "gauge transformations!")

- Start with $\nabla\cdot\vec{B}=0$.
  Therefore we can write $\vec{B}=\nabla\times\vec{A}$.

[Because $\nabla\cdot(\nabla\times\vec{A}) = \frac{\partial B_i}{\partial x_i} = \frac{\partial}{\partial x_i}\epsilon_{ijk}\frac{\partial}{\partial x_j}A_k$

$$\underbrace{\epsilon_{ijk}}_{\text{antisymmetric in } i,j}\ \underbrace{\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}}_{\text{symmetric in } i,j} A_k = 0, \text{ automatically}$$

$$= \frac{\partial}{\partial x}\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial y}\frac{\partial}{\partial x}A_z + 4 \text{ other terms} = 0$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} \quad \text{identically}$$

$$\hookrightarrow \lim_{\epsilon_x\to 0}\lim_{\epsilon_y\to 0}\frac{\{f(x+\epsilon_x,y+\epsilon_y,z)-f(x,y+\epsilon_y,z)-f(x+\epsilon_x,y,z)+f(x,y,z)\}}{\epsilon_x\,\epsilon_y}$$

same for either order!

$\boxed{\text{Partial derivatives commute.}}$]

$\vec{A}$ is not unique, but definitely $\exists\,\vec{A}$ s.t. $\vec{B}=\nabla\times\vec{A}$.
That guarantees $\nabla\cdot\vec{B}=0$.

Next, we require $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$.

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and $\vec{B} = \nabla \times \vec{A}$ R 9/3

$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$

$\left\{ \nabla \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \nabla \right.$
Partial derivatives commute.

Therefore we can write

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$$

Because $\nabla \times (\nabla V) = \hat{e}_i \underbrace{\epsilon_{ijk}}_{\text{anti-symmetric}} \underbrace{\frac{\partial}{\partial x_j} \cdot \frac{\partial}{\partial x_k}}_{\text{symmetric}} V = 0.$

$= \hat{i}\left(\frac{\partial}{\partial y}\frac{\partial}{\partial z} V - \frac{\partial}{\partial z}\frac{\partial}{\partial y} V \right) + 4$ other terms

$= 0$

/partial derivatives commute/

$V(\vec{x},t)$ is not unique, but $\exists V$ s.t. $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$.

$$\boxed{\begin{aligned} \vec{B} &= \nabla \times \vec{A} \\ \vec{E} &= -\nabla V - \frac{\partial \vec{A}}{\partial t} \end{aligned}}$$

So far we have no equations to determine $\vec{A}(\vec{x},t)$ and $V(\vec{x},t)$. The other Maxwell equations determine $\vec{A}$ and V

$\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ R 0/4

The potentials, $\vec{A}(\vec{x},t)$ and $V(\vec{x},t)$, must be "determined" by the other Maxwell equations; dependent on $\rho(\vec{x},t)$ and $\vec{J}(\vec{x},t)$.

$$\nabla \cdot \vec{E} = \rho/\epsilon_0 \Rightarrow \quad -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = \frac{\rho}{\epsilon_0} \quad (1)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Rightarrow$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$= \mu_0 \vec{J} + \frac{1}{c^2}\left\{ -\nabla \frac{\partial V}{\partial t} - \frac{\partial^2 \vec{A}}{\partial t^2} \right\}$$ (Partial derivatives commute.)

$$-\nabla^2 \vec{A} + \nabla(\nabla \cdot \vec{A}) + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla V = \mu_0 \vec{J} \quad (2)$$

Equations (1) and (2) give 2 equations for 2 unknowns ($\vec{A}(\vec{x},t)$ and $V(\vec{x},t)$).

The solutions are not unique, but we'll be able to construct solutions.

Gauge Transformations

Ro/5

$\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$.

The potentials are not unique.

For any scalar function, $\lambda(\vec{x},t)$, define

$$\vec{A}'(\vec{x},t) = \vec{A}(\vec{x},t) - \nabla\lambda$$
$$V'(\vec{x},t) = V(\vec{x},t) + \frac{\partial \lambda}{\partial t}$$

} called the gauge transformation.

Then

$$\nabla \times \vec{A}' = \nabla \times \vec{A} + 0 = \vec{B}$$

and

$$-\nabla V' - \frac{\partial \vec{A}'}{\partial t} = -\nabla V - \nabla \frac{\partial \lambda}{\partial t} - \frac{\partial \vec{A}}{\partial t} + \frac{\partial}{\partial t}\nabla\lambda$$
$$= \vec{E}$$

cancel (partial derivatives commute)

I.e., the potentials $(\vec{A}', V')$ describe the same fields $(\vec{E}, \vec{B})$ as $(\vec{A}, V)$. The potential functions are not unique.

Gauge transformations

$$\boxed{\begin{aligned} \vec{A}' &= \vec{A} - \nabla\lambda \\ V' &= V + \frac{\partial \lambda}{\partial t} \end{aligned}}$$

R0/6

It follows that Eqs. (1) and (2) are also satisfied by $\vec{A}'$ and V'! (Derive from same $\vec{E}$ & $\vec{B}$.)

Verify directly ...

$V' = V + \frac{\partial\lambda}{\partial t}$ and $\vec{A}' = \vec{A} - \nabla\lambda$

(1) • $-\nabla^2 V' - \frac{\partial}{\partial t}(\nabla \cdot \vec{A}')$

$$= -\nabla^2 V - \nabla^2 \frac{\partial \lambda}{\partial t} - \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) + \frac{\partial}{\partial t}\nabla^2\lambda = \rho/\epsilon_0 \checkmark$$

cancel

because partial derivatives commute

(2) • $-\nabla^2 \vec{A}' + \nabla(\nabla \cdot \vec{A}') + \frac{1}{c^2}\frac{\partial^2 \vec{A}'}{\partial t^2} + \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla V')$

$$= -\nabla^2 \vec{A} + \nabla(\nabla \cdot \vec{A}) + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} + \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla V) \Big\} = \mu_0 \vec{J}$$

$$+\nabla^2(\nabla\lambda) - \nabla(\nabla^2\lambda) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\nabla\lambda) + \frac{1}{c^2}\frac{\partial \nabla}{\partial t}\left(\frac{\partial \lambda}{\partial t}\right)\Big\} \rightarrow 0$$

because partial derivatives commute

$$= \mu_0 \vec{J}$$

Gauge Transformations $\quad \vec{A}' = \vec{A} - \nabla\lambda$

$$V' = V + \frac{\partial\lambda}{\partial t}$$

where $\lambda(\vec{x},t)$ is arbitrary.

Gauge Condition, or gauge choice R o/7

. Because $(\vec{A}, V)$ is not uniquely determined, we can impose another condition to fix $(\vec{A}, V)$. "gauge fixing"

- Coulomb gauge Require $\nabla \cdot \vec{A} = 0$.

Then (1) : $-\nabla^2 V = \rho/\epsilon_0$

(2) : $-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J} - \frac{1}{c^2}\nabla\left(\frac{\partial V}{\partial t}\right)$

- Lorentz gauge Require $\nabla \cdot \vec{A} = \frac{-1}{c^2}\frac{\partial V}{\partial t}$

Then (1) : $-\nabla^2 V + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \frac{\rho}{\epsilon_0}$

(2) : $-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}$

which is used in the theory of radiation.

NOW READ CHAPTER 15.