Radiation of electromagnetic waves

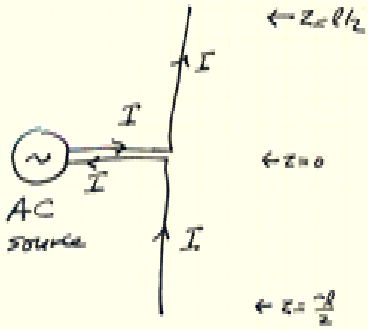
⇒ Technology of radio communication

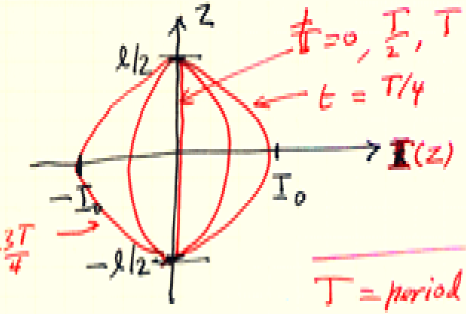
1937 Funeral of Marconi

Le

The half-wave linear antenna

R3/1





$T = \text{period} = \frac{2\pi}{\omega}$

"center-fed linear antenna" $\Rightarrow I(z,t)$

$I(0,t) = I_0 \sin\omega t$ and $I(\pm\frac{\ell}{2}, t) = 0$.

For the "half-wave antenna," $\ell = \frac{\lambda}{2}$.

Then $I(z,t)$ looks like the graph above....

$$I(z,t) \approx I_0 \cos(kz)\sin(\omega t)$$

Boundary Condition: $\cos(k\ell/2) = 0$

period $T = 2\pi/\omega$; length $\ell = \frac{\pi}{k}$ ⟵ $k\frac{\ell}{2} = \pi/2$

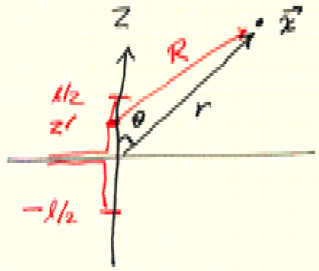
[Note that k and ω are related (∗). We'll see that this is necessary to have a self-consistent field solution.
For $\ell \neq \lambda/2$ we'd also need to determine $I(z,t)$ self consistently]

(∗) $\lambda = \frac{c}{f} = \frac{2\pi c}{\omega}$ and $\lambda = 2\ell = \frac{2\pi}{k}$

$\therefore \omega = ck$.

$$\vec{A}(\vec{x},t) = \mu_0 \int \frac{\vec{J}(\vec{x}', t - R/c)}{4\pi R} d^3x' \quad \text{where } R = |\vec{x} - \vec{x}'|$$

$$\vec{A}(\vec{x},t) = \mu_0 \int_{-\ell/2}^{\ell/2} \frac{I(z', t - R/c)\, dz'}{4\pi R} \hat{k}$$



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$$\vec{A}(\vec{x},t) = \mu_0 \int_{-\ell/2}^{\ell/2} \frac{I(z', t - R/c)\,dz'}{4\pi R}\;\hat{k}$$

$$R = \sqrt{r^2 - 2rz'\cos\theta + z'^2}$$

$$R \approx r - z'\cos\theta \quad \text{or} \quad R \approx r.$$

$\vec{R} = \vec{x} - \vec{x}\,'$

$R = |\vec{x} - \vec{x}\,'|$

$$\vec{A}(\vec{x},t) \sim \frac{\mu_0}{4\pi r}\int_{-\ell/2}^{\ell/2} I\left(z', t - \frac{r}{c} + \frac{z'}{c}\cos\theta\right) dz'\;\hat{k}$$

$$\vec{A}_{rad} = \frac{\mu_0 I_0}{4\pi r}\underbrace{\int_{-\ell/2}^{\ell/2} \cos(kz')\sin\left(\omega t - \frac{\omega r}{c} + \frac{\omega z'}{c}\cos\theta\right) dz'}_{\mathcal{J}}\;\hat{k}$$

$\frac{\omega}{c} = k$

The integral $\mathcal{J}$

Let $\psi = kz'$. Then $d\psi = k\,dz'$. $\quad k = \frac{\pi}{\ell}$

End points : $z' = \pm \ell/2 \;\Rightarrow\; \psi = \pm\frac{k\ell}{2} = \pm\frac{\pi}{2}$

$$\mathcal{J} = \frac{1}{k}\int_{-\pi/2}^{\pi/2} \cos\psi \,\sin(\psi\gamma - g)\, d\psi$$

where $\gamma = \cos\theta$ and $g = kr - \omega t$

$$= \frac{1}{k}\int_0^{\pi/2} \cos\psi\left[\sin(\psi\gamma - g) - \sin(\psi\gamma + g)\right] d\psi$$

$$= \frac{-2\sin g}{k}\int_0^{\pi/2} \cos\psi\,\cos(\psi\gamma)\, d\psi$$

$$= \frac{-2\sin g}{k}\int_0^{\pi/2}\left\{\cos(\psi - \psi\gamma) + \cos(\psi + \psi\gamma)\right\}\frac{d\psi}{2}$$

$$= -\frac{\sin g}{k}\left\{\frac{1}{1-\gamma}\sin(\psi - \psi\gamma) + \frac{1}{1+\gamma}\sin(\psi + \psi\gamma)\right\}\Bigg|_{\psi=0}^{\pi/2}$$

$$= \frac{-\sin g}{k}\left\{\frac{1}{1-\gamma}\cos\frac{\pi\gamma}{2} + \frac{1}{1+\gamma}\cos\frac{\pi\gamma}{2}\right\}$$

R3/3

$$\therefore \quad g = -\frac{\sin q}{k}\,\frac{2}{1-\gamma^2}\,\cos\frac{\pi\gamma}{2}$$

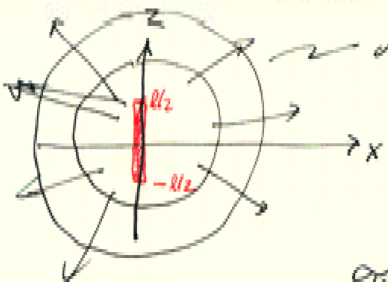
Remember: $\gamma = \cos\theta$

$q = kr - \omega t$

& $\vec{A}_{rad} = \frac{\mu_0 I_0}{4\pi r}\, g\, \hat{k}$

So

$$\vec{A}_{rad} = \frac{-2\mu_0 I_0}{4\pi kr}\,\frac{\cos\left[\frac{\pi}{2}\cos\theta\right]}{\sin^2\theta}\,\sin(kr-\omega t)\quad\hat{k}$$



outgoing electromagnetic waves

Spherical waves

$$\vec{A} \propto \frac{\sin(kr-\omega t)}{kr}\,\hat{k}$$

The wave fronts are spheres $kr - \omega t = \frac{n\pi}{2}$

Phase velocity $= \frac{\omega}{k}$ because $k\,\delta r - \omega\,\delta t = 0$

$$\frac{\delta r}{\delta t} = \frac{\omega}{k}$$

The phase velocity must be c, to be consistent with Maxwell's equations;

so $\omega = ck \longleftarrow$ the condition $l = \frac{\lambda}{2}$. because $\frac{kl}{2} = \frac{\pi}{2}$, $kl = \pi$

Also, $\lambda = \frac{2\pi}{k}$ so $k = \frac{\pi}{l}$ ✓. $\longrightarrow \omega = ck$

$$\vec{A}_{rad} = \frac{-\mu_0 I_0}{2\pi}\,\frac{\cos\left[\frac{\pi}{2}\cos\theta\right]}{\sin^2\theta}\,\frac{\sin(kr-\omega t)}{kr}\quad\hat{k}$$

$\vec{A}_{rad} = A_{rad}\,\hat{k}$

The energy flux

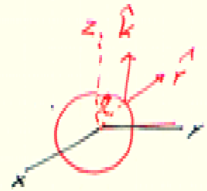
R3/4

$$\vec{B}_{rad} = \nabla\times\vec{A}_{rad} = \hat{e}_i\,\epsilon_{ijk}\frac{\partial}{\partial x_j}A_{rad}\,\delta_{k3}$$

$$= \nabla A_{rad}\times\hat{k}$$

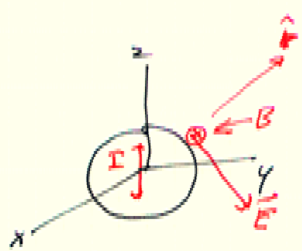
$\nabla\sin(kr-\omega t) = \hat{r}\,k\cos(kr-\omega t)$ is the asymptotic gradient (neglecting terms of order $1/r^2$)

$$\vec{B}_{rad} = \frac{-\mu_0 I_0}{2\pi r}\left\{\begin{matrix}\cos/\sin^2\\ (f(\theta))\\ \text{func of } \theta\end{matrix}\right\}\cos(kr-\omega t)\,\underbrace{\hat{r}\times\hat{k}}_{-\hat{\phi}\sin\theta}$$



$$\vec{B}_{rad} = \frac{\mu_0 I_0}{2\pi r}\,f(\theta)\cos(kr-\omega t)\,\hat{\phi}$$

$$\text{where}\quad f(\theta) = \frac{\cos\left[\frac{\pi}{2}\cos\theta\right]}{\sin\theta}$$



$$\vec{E}_{rad} = c\,\vec{B}_{rad}\times\hat{r}$$

$$= c\,B_{rad}\;\hat{\phi}\times\hat{r}$$

$$= c\,B_{rad}\;\hat{\theta}$$

$\vec{E}, \vec{B}, \hat{r}$ form an orthogonal triad in the radiation zone

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{c}{\mu_0}B_{rad}^2\;\hat{\theta}\times\hat{\phi} = \frac{c}{\mu_0}B_{rad}^2\;\hat{r}$$

- The energy flows radially away from the antenna (in the radiation zone).
- $\dfrac{dP_{avg}}{d\Omega} = r^2\,\hat{r}\cdot\langle\vec{S}\rangle$ where $\langle\cos^2(kr-\omega t)\rangle = 1/2$ (time average)

$$\frac{dP_{avg}}{d\Omega} = \frac{c}{\mu_0}\left(\frac{\mu_0 I_0}{2\pi}\right)^2 f^2(\theta)\,\frac{1}{2} = \frac{\mu_0 c I_0^2}{8\pi^2}\,f^2(\theta)$$

units : $\frac{Tm}{A}\,\frac{m}{s}\,A^2 = \frac{Vs}{m^2}\,\frac{m^2}{s}\,A = W$

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Comparing half-wave antenna and Hertzian dipole

HW antenna
$$\frac{dP_{avg}}{d\Omega} = \frac{\mu_0 c I_0^2}{8\pi^2} f^2(\theta)$$

where $f(\theta) = \dfrac{\cos\left[\frac{\pi}{2}\cos\theta\right]}{\sin\theta}$

Recall the Hertzian dipole

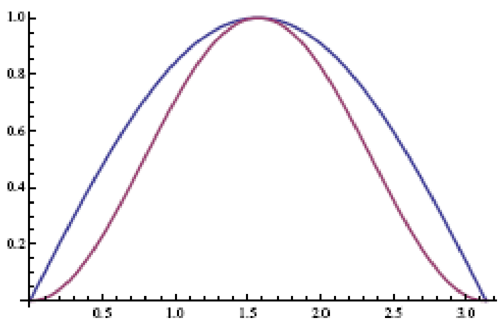
$$\frac{dP_{avg}}{d\Omega} = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \sin^2\theta$$

- Compare the angular (θ) dependences $f(\theta)$, $f^2(\theta)$; $\sin\theta$, $\sin^2\theta$
- Compare the angular distributions of intensity, $dP_{avg}/d\Omega$ versus direction

Hertzian Dipole

```
In[45]:= fHD[th_] := Sin[th]
         Plot[{fHD[th], fHD[th]^2}, {th, 0, Pi},
          PlotStyle -> AbsoluteThickness[2]]
```

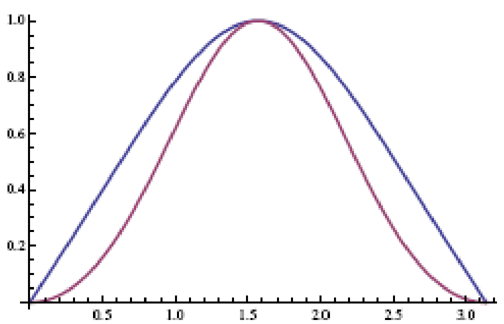
Out[46]=



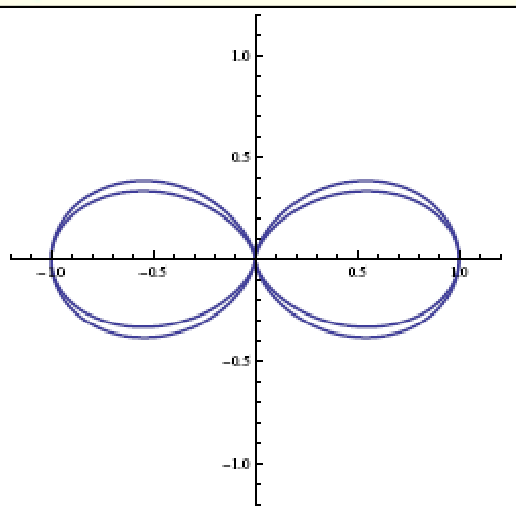
Half – wave

```
In[13]:= fHW[th_] := Cos[Pi / 2 * Cos[th]] / Sin[th]
         Plot[{fHW[th], fHW[th]^2}, {th, 0, Pi},
          PlotStyle -> AbsoluteThickness[2]]
```

Out[14]=



Out[129]=



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Total power

Hertzian Dipole $\quad \dfrac{dP}{d\Omega} = \dfrac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \sin^2\theta$

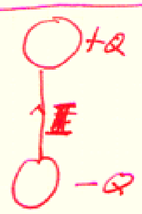
$$P = \int \frac{dP}{d\Omega} \sin\theta \, d\theta \, d\varphi = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \cdot \frac{4}{3} \cdot 2\pi$$

$$P = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$

Here : $p_0 = Q_0 d$

$$I = \dot{Q} = \omega Q_0 \sin\omega t$$

$$\langle I^2 \rangle = \frac{\omega^2 Q_0^2}{2} = (I_{rms})^2$$



$\boxed{\text{ANTENNA RESISTANCE} \quad P_{av} = I_{RMS}^2 \cdot R_{ant}}$

$$R_{ant} = \frac{P_{av}}{I_{rms}^2} = \frac{\mu_0 Q_0^2 \omega^4 d^2}{12\pi c \, \frac{1}{2}\omega^2 Q_0^2} = \frac{1}{6\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}(kd)^2$$

$\sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega \qquad$ so $\quad R_{ant} = (20\,\Omega)(kd)^2 \ll 20\,\Omega$

for the Hertzian dipole.

Half-wave antenna $\quad \dfrac{dP}{d\Omega} = \dfrac{\mu_0 c I_0^2}{8\pi^2} f^2(\theta)$ where $f(\theta) = \dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$

$$P = \frac{\mu_0 c I_{rms}^2}{4\pi^2} \underbrace{\int f^2(\theta) \sin\theta \, d\theta}_{1,22} \, \underbrace{d\varphi}_{2\pi} = \frac{1,22}{2\pi}\sqrt{\frac{\mu_0}{\epsilon_0}} I_{rms}^2$$

$R_{ant} = \dfrac{1.22}{2\pi}\sqrt{\dfrac{\mu_0}{\epsilon_0}} = 73\,\Omega$ for the half-wave antenna

Quiz Question

You have probably observed that radio towers are tall.
How tall?
Calculate the height of a half-wave linear antenna if the frequency is 1 MHz.