# The field $\vec{H}(\vec{x})$ ; Ampère's law in matter 

The fundamental field equation that relates

current density $\vec{J}(\vec{x})$ [units: $C/m^2/s = A/m^2$]

and magnetic field $\vec{B}(\vec{x})$ [T]

is $\nabla \times \vec{B} = \mu_0 \vec{J}$ or $\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{through}}$

But to be practical, we must approximate the microscopic bound current:

$$\vec{J}_B = \nabla \times \vec{M} \quad \text{where} \quad \vec{M}(\vec{x}) = n(\vec{x}) \langle \vec{m} \rangle_{AVG}$$

$$\vec{J}(\vec{x}) = \vec{J}_{\text{Free}}(\vec{x}) + \vec{J}_{\text{Bound}}(\vec{x})$$

Free current in the system

Current bound in the atoms, averaged over a small volume

$$\nabla \times \vec{B} = \mu_0 \vec{J}_F + \mu_0 \nabla \times \vec{M}$$

$$\nabla \times \left( \frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_F$$

$$\boxed{\text{Define } \vec{H}(\vec{x}) = \frac{\vec{B}}{\mu_0} - \vec{M} \text{ ; then } \nabla \times \vec{H} = \vec{J}_{\text{Free}}}$$

(Recall displacement $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ and $\nabla \cdot \vec{D} = \rho_{\text{Free}}$.)

<u>Nomenclature</u> (not standardized)

Flux density — $\vec{B}$ = fundamental magnetic field ; $\vec{F} = q \vec{v} \times \vec{B}$

Field strength — $\vec{H}$ = magnetizing field

## Ampère's law

$$\nabla \times \vec{H} = \vec{J}_{Free} \qquad \text{where} \quad \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

The integral form

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S (\nabla \times \vec{H}) \cdot d\vec{A} = \int_S \vec{J}_F \cdot d\vec{A} = I_{F,\,\text{through}}$$

Stokes's theorem ↗     ↑ Ampère's law

$$\oint_C \vec{H} \cdot d\vec{l} = I_{Free} \ \text{(through } C\text{)}$$

If the form of $\vec{H}(\vec{x})$ is known by symmetry, we may be able to determine $\vec{H}(\vec{x})$ using the integral.

## Linear Materials

Isotropic ; diamagnetic or paramagnetic $\Longrightarrow$

$$\vec{M}(\vec{x}) \propto \vec{H}(\vec{x})$$

"The magnetization is proportional to the magnetizing field."

Material properties

- Susceptibility $\chi_m$ is defined by $\vec{M} = \chi_m \vec{H}$.
- Permeability $\mu$ is defined by $\vec{B} = \mu \vec{H}$.

They are related :

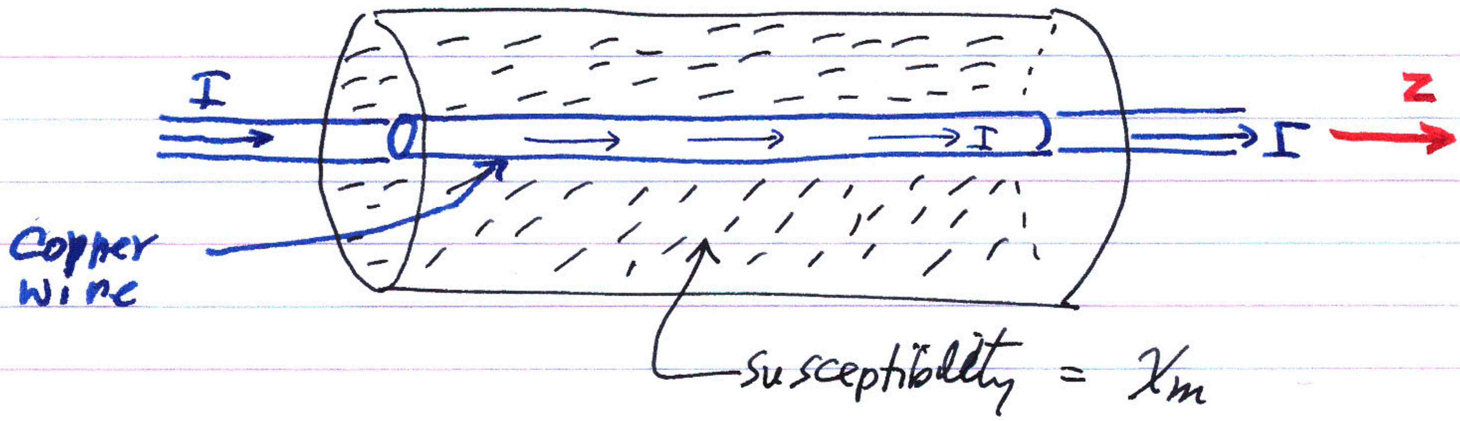
$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad \Longrightarrow \quad \mu_0 (1 + \chi_m) \vec{H} = \vec{B}$$

Thus $\qquad \mu = \mu_0 (1 + \chi_m)$

$\mu_0 = 4\pi \times 10^{-7}$ Tm/A , universal constant

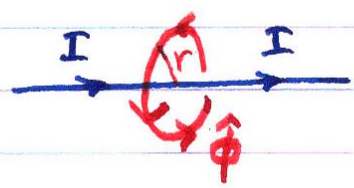
$\chi_m$ : look it up in a table.

# Example. A long wire embedded in magnetic material




(a) What is the magnetic field?

By symmetry, $\vec{H}(\vec{x}) = H(r)\,\hat{\phi}$




Consider a circular Amperian loop

$$\oint_C \vec{H}\cdot d\vec{\ell} = H(r)\cdot 2\pi r = I \quad \text{(free current)}$$

$$H(r) = \frac{I}{2\pi r}$$

($r$ = $\perp$ distance to the wire)

$$\vec{B}(\vec{x}) = \frac{\mu I}{2\pi r}\,\hat{\phi}$$

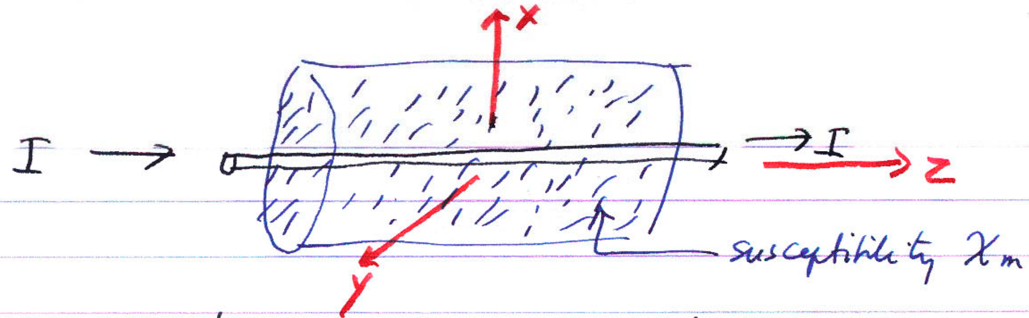
$$\vec{M} = \chi_m \vec{H}$$
$$\mu = \mu_0(1+\chi_m)$$

- Paramagnetic material:

  $\chi_m > 0$, $\mu > \mu_0$, $B > B_{vacuum}$

- Diamagnetic material:

  $\chi_m < 0$, $\mu < \mu_0$, $B < B_{vacuum}$




(b) What is the Magnetization?

$$\vec{M}(\vec{x}) = \chi_m \vec{H} = \frac{\chi_m I}{2\pi r}\,\hat{\phi}$$

$$\vec{M} = \frac{\chi_m I}{2\pi\sqrt{x^2+y^2}}\left(-\frac{y}{r}\hat{\imath} + \frac{x}{r}\hat{\jmath}\right) = \frac{\chi_m I}{2\pi}\,\frac{(-y\hat{\imath} + x\hat{\jmath})}{x^2+y^2}$$

(c) What is the Bound current density?

- Volume density $\vec{J}_B = \nabla\times\vec{M}$

$$= \nabla\times(\chi_m\vec{H}) = \chi_m \nabla\times\vec{H} = 0$$

Note: $\vec{J}_{free} = 0$ in the material

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y/r^2 & x/r^2 & 0 \end{vmatrix} = \hat{k}\left[\frac{1}{r^2} + \frac{x(-2x/r)}{r^3} + \frac{1}{r^2} + \frac{y(-2y/r)}{r^3}\right]$$

$r^2 = x^2+y^2$ $\qquad = 0$

- Surface density, i.e., on the surface of the copper wire,

$$\vec{K}_b = \hat{M}\times\hat{n} = \frac{\chi_m I}{2\pi a}\,\hat{\phi}\times(-\hat{r})$$

outward normal (out of the material)

a = radius of copper wire

$$\vec{K}_b = \frac{\chi_m I}{2\pi a}\,\hat{k}$$

$\Bigg\{$ parallel $\vec{I}$ for paramagnetic material

antiparallel $\vec{I}$ for diagmagnetic material

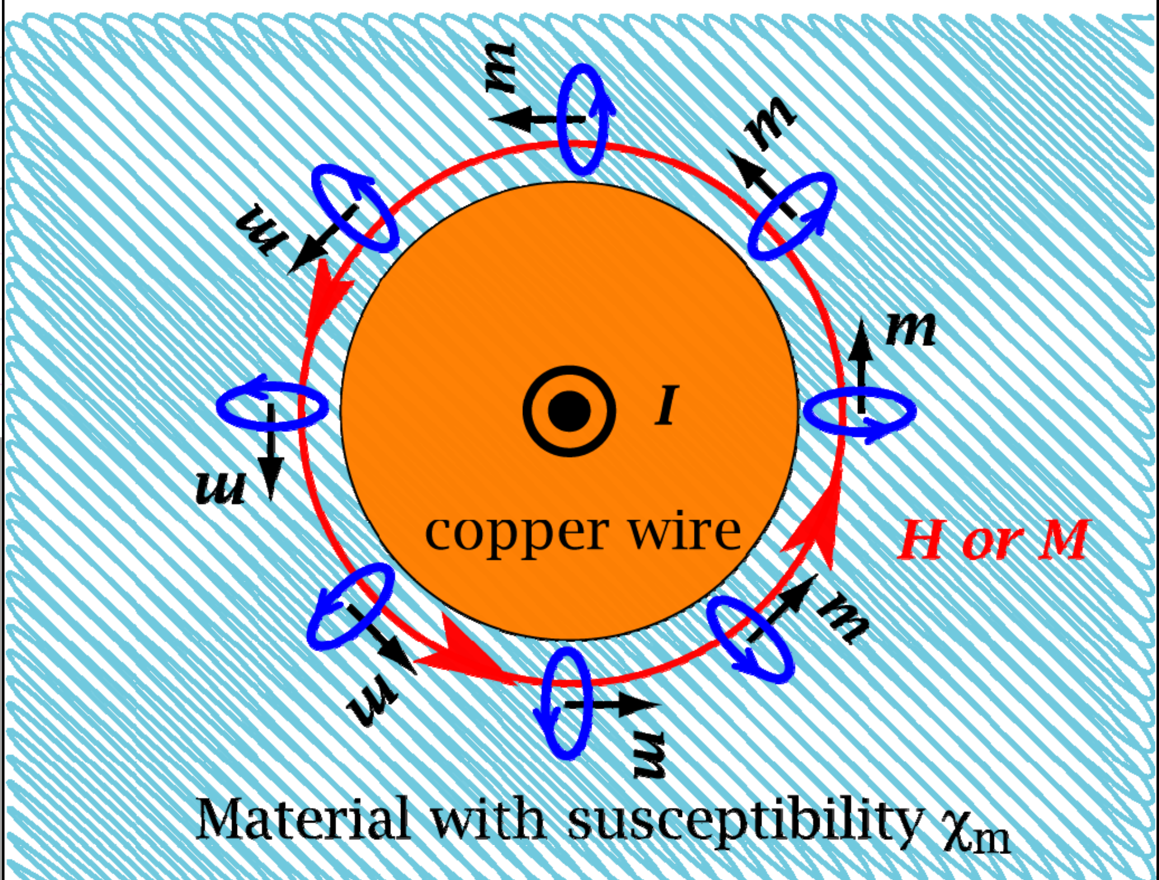
An interesting question of geometry 9.3/5

$$\vec{M} = \frac{\chi_m I}{2\pi r}\hat{\phi} \quad \propto \quad \frac{\hat{\phi}}{r}$$

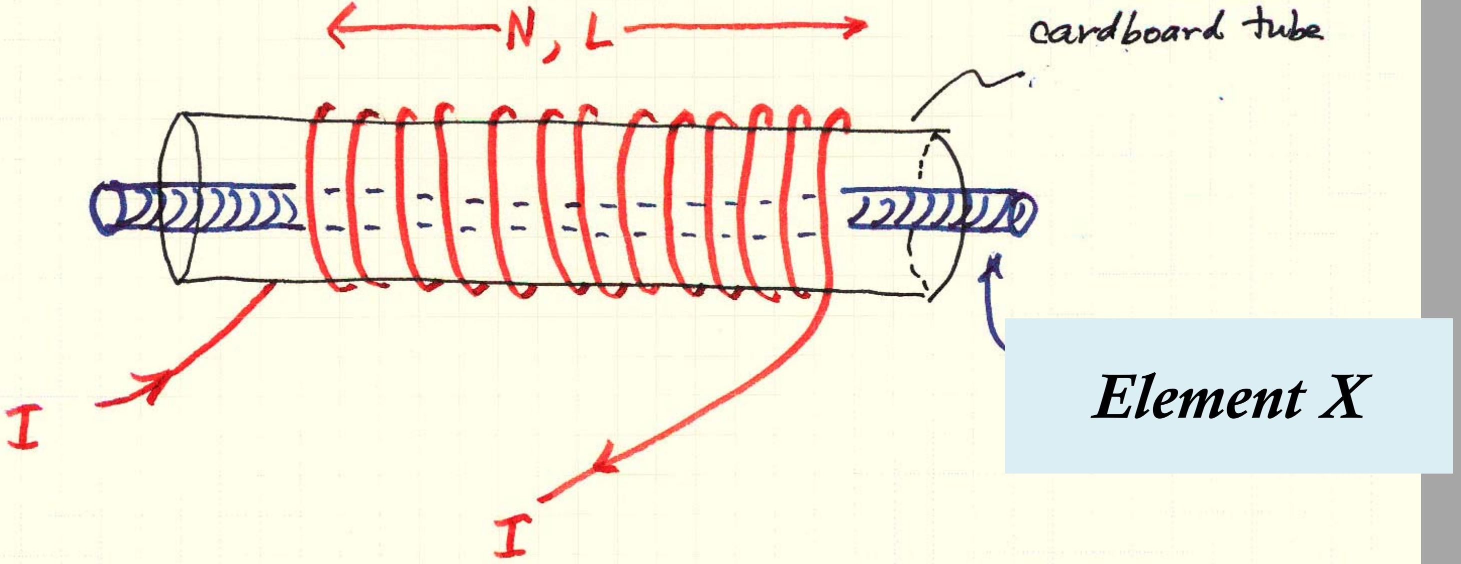
$$\vec{K}_b = \frac{\chi_m I}{2\pi a}\hat{k} \quad \propto \quad \hat{k}$$

Why does an azimuthal magnetization produce an axial surface current?

A copper wire with current $I$ is embedded in a magnetic material; end view:




Note the *bound surface current*, pointing out of the picture, at the surface of the copper wire.




Parameters: $N = 1000$, $L = 0.1\ m$

$I = 3\ A$

(a) Calculate $\vec{B}$ inside the solenoid but outside the Element X

(b) Calculate $\vec{B}$ in the Element X