Boundary Conditions for $\vec{B}$ and $\vec{H}$

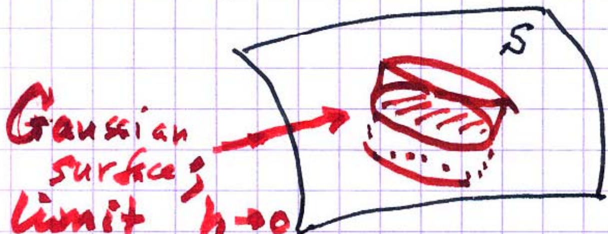
Result (1) B_n is continuous.

$$B_n^{(+)} = B_n^{(-)} \quad \text{on any surface}$$

(2) H_t is continuous, unless there is a Free surface current, in which case

$$H_t^{(+)} - H_t^{(-)} = K_{free}$$

Proof (1)



$\nabla \cdot \vec{B} = 0$ ← a fundamental equation which says that magnetic monopoles do not Exist.

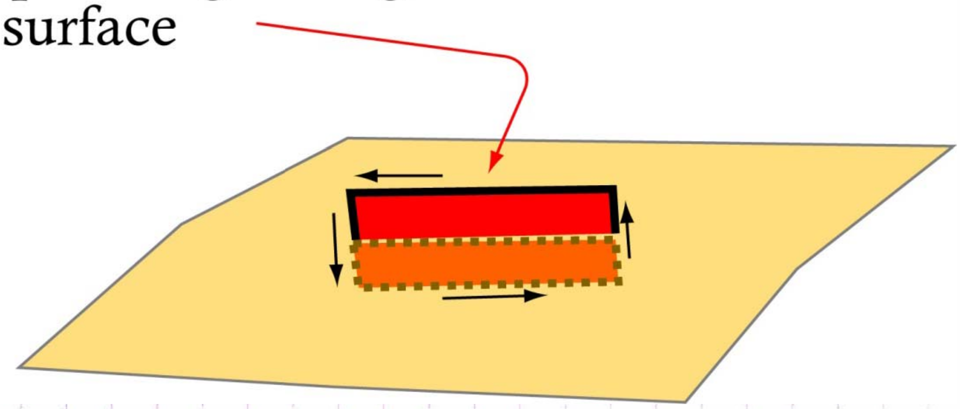
$$\int \nabla \cdot \vec{B}\, d^3x = \oint \vec{B} \cdot d\vec{A}$$

$$= B_n^{(+)} A - B_n^{(-)} A = 0$$

$$\lim_{h \to 0} \left[B_n^{(+)} - B_n^{(-)} \right] = 0$$

i.e. B_n is continuous Q.E.D.

Proof (2)



$$\nabla \times \vec{H} = \vec{J}_{free} \implies \oint_C \vec{H} \cdot \vec{dl} = I_{f\,\text{through}}$$

$$\lim_{h \to 0} \oint \vec{H} \cdot \vec{dl} = H_t^{(+)} \cdot \vec{\delta l} - H_t^{(-)} \cdot \vec{\delta l}$$

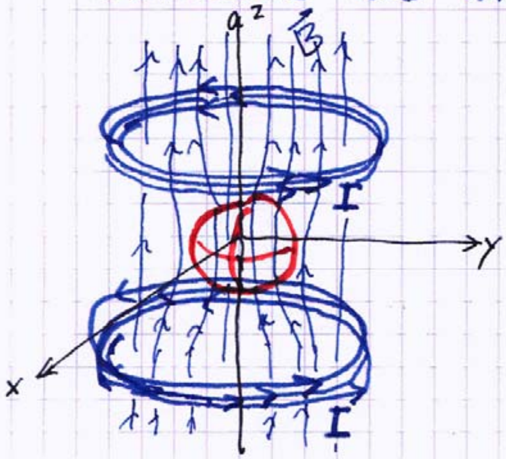
$$= I_F = \underbrace{\vec{K}_F \cdot (\hat{n} \times \vec{\delta l})}_{= (\vec{K}_F \times \hat{n}) \cdot \vec{\delta l}}$$

a cross product identity

$$\lim_{h \to 0} \left\{ H_t^{(+)} - H_t^{(-)} \right\} = \vec{K}_{Free} \times \hat{n}$$

Q.E.D.

Example. A solid sphere, of a material with magnetic susceptibility χ_m, is placed in a uniform applied magnetic field. Determine the magnetic field.



Helmholtz coils produce approximately uniform field.

Sphere has radius b, susceptibility χ_m.

The asymptotic field $(r \gg b)$ is $B_0 \hat{k}$

Field Equations

- Outside the sphere $(r > b)$

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{H} = 0 \quad \text{and} \quad \vec{B} = \mu_0 \vec{H}$$

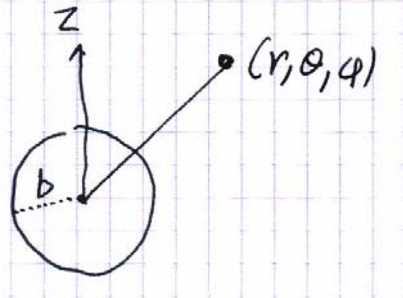
- Inside the sphere $(r < b)$

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{H} = 0 \quad \text{and} \quad \vec{B} = \mu \vec{H}$$
$$(\star) \qquad\qquad (\star\star)$$

So $\vec{H}$ and $\vec{B}$ are completely determined by the BOUNDARY CONDITIONS

$(\star)$ always true

$(\star\star)$ there is no free current except the Helmholtz coils, which are far away ("at ∞")



Solve by Laplace's Equation

Because $\nabla \times \vec{H} = 0$ we can write $\vec{H} = -\nabla \phi_m$

(magnetic scalar potential)

Then $\nabla \cdot \vec{H} = 0$ implies $\nabla^2 \phi_m = 0$

$$\nabla \cdot \vec{H} = \nabla \cdot \left(\vec{B}/\mu \right) = \frac{\nabla \cdot \vec{B}}{\mu} = 0 \quad (\mu_0 \text{ or } \mu)$$

So ϕ_m obeys Laplace's equation in both regions.

The asymptotic potential must be

$\phi_m \sim -\frac{B_0}{\mu_0} z$ so that $-\nabla \phi_m \sim \frac{B_0 \hat{k}}{\mu_0}$

$\therefore \phi_m \sim -\frac{B_0}{\mu_0} r \cos\theta.$

That suggests that we should try solutions with $\phi_m \propto \cos\theta$. Guess...

$$\phi_m(r, \theta) = \begin{cases} \alpha r \cos\theta & \text{for } r < b \quad \text{(inside)} \\ -\frac{B_0}{\mu_0} r \cos\theta + \beta \frac{\cos\theta}{r^2} & \text{for } r > b \quad \text{(outside)} \end{cases}$$

(These functional forms obey Laplace's equation)

$$\phi_m = \begin{cases} \alpha r\cos\theta & \text{for } r<b \\ -\dfrac{B_0}{\mu_0} r\cos\theta + \beta\dfrac{\cos\theta}{r^2} & \text{for } r>b \end{cases}$$

BOUNDARY CONDITIONS at $r=b$ 9.4/5

$$\boxed{\vec{H} = -\nabla\phi_m}$$

☆ B_r is continuous

$$-\mu\frac{\partial\phi_m^{(-)}}{\partial r} = -\mu_0\frac{\partial\phi_m^{(+)}}{\partial r} \quad \text{at } r=b$$

$$-\mu\,\alpha\cos\theta = -\mu_0\left[\left(\frac{-B_0}{\mu_0}\right)\cos\theta - \frac{2\beta}{b^3}\cos\theta\right]$$

$$-\mu\alpha = B_0 + \frac{2\mu_0\beta}{b^3} \quad (1)$$

☆ H_θ is continuous (because there is no free current)

$$-\frac{1}{r}\frac{\partial\phi_m^{(-)}}{\partial\theta} = \frac{-1}{r}\frac{\partial\phi_m^{(+)}}{\partial\theta} \quad \text{at } r=b$$

$$\alpha\, b\sin\theta = \frac{-B_0}{\mu_0}b\sin\theta + \frac{\beta}{b^2}\sin\theta$$

$$\alpha = -\frac{B_0}{\mu_0} + \frac{\beta}{b^3} \quad (2)$$

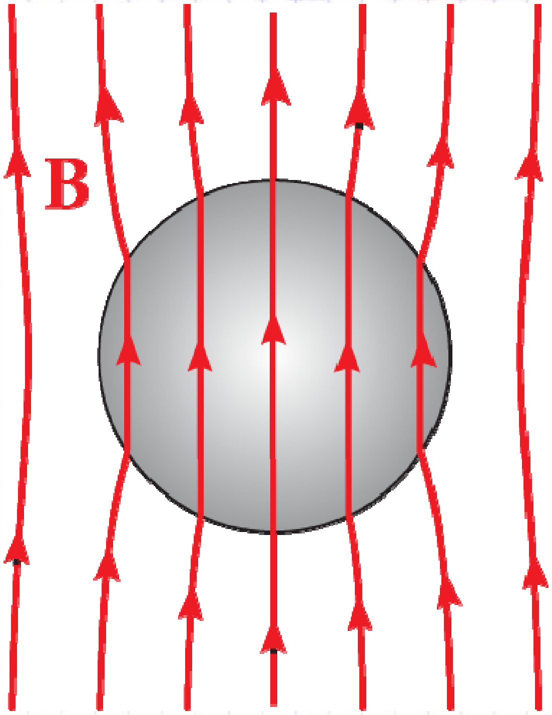
Solve for α and β

Express the answers in terms of $K_m = \mu/\mu_0$

$$\alpha = \frac{-3}{K_m+2}\frac{B_0}{\mu_0} \quad \text{and} \quad \beta = \frac{K_m-1}{K_m+2}\frac{b^3 B_0}{\mu_0}$$

<u>Solution</u>

$$\phi_m = \begin{cases} \alpha r\cos\theta & \text{for } r<b \\ -\dfrac{B_0}{\mu_0} r\cos\theta + \beta\dfrac{\cos\theta}{r^2} & \text{for } r>b \end{cases}$$



$$\alpha = \frac{-3}{K_m+2}\,\frac{B_0}{\mu_0}$$

$$\beta = \frac{K_m-1}{K_m+2}\,\frac{b^3 B_0}{\mu_0}$$

$$K_m = \frac{\mu}{\mu_0} = 1+\chi_m$$

$$\vec{H} = -\nabla\phi_m \quad \text{and} \quad \vec{B} = \begin{cases} \mu\vec{H} \\ \mu_0\vec{H} \end{cases}$$

Paramagnetic case

<u>Special Cases</u>

- $K_m = 1$ implies $\phi_m = -\dfrac{B_0}{\mu_0} r\cos\theta$ ← $\chi_m = 0$ no magnetization
  i.e., a uniform field ; $\vec{B} = B_0\hat{k}$.
- $K_m = 0$ implies $\vec{B} = 0$ inside the material ; $\vec{B}_{ext} = B_0\hat{k} - \dfrac{b^3 B_0}{2r^3}\left[3\hat{r}\cos\theta - \hat{k}\right]$ ← $\chi_m = -1$ perfect diamagnet
- $K_m \gg 1$ implies ← $\chi_m \gg 1$ strong paramagnet or ferromagnet
  $\vec{B}_{int} = 3B_0\hat{k}$
  $\vec{B}_{ext} = B_0\hat{k} + K_m\dfrac{b^3 B_0}{r^3}\left[3\hat{r}\cos\theta - \hat{k}\right]$

[Diamagnetic case: Levitating Frog]