Annalen der Physik, June 1905

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3. *Zur Elektrodynamik bewegter Körper;* *von A. Einstein.*

Daß die Elektrodynamik Maxwells — wie dieselbe gegenwärtig aufgefaßt zu werden pflegt — in ihrer Anwendung auf bewegte Körper zu Asymmetrien führt, welche den Phänomenen nicht anzuhaften scheinen, ist bekannt. Man denke z. B. an

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$\varphi(v) = 1$ sein muß, so daß die gefundenen Transformationsgleichungen übergehen in:

$$\tau = \beta\left(t - \frac{v}{V^2}x\right),$$

$$\xi = \beta(x - vt),$$

$$\eta = y,$$

$$\zeta = z,$$

wobei

$$\beta = \frac{1}{\sqrt{1 - \left(\frac{v}{V}\right)^2}},$$

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und unsere Gleichungen nehmen die Form an:

$$X' = X, \qquad L' = L,$$

$$Y' = \beta\left(Y - \frac{v}{V}N\right), \qquad M' = \beta\left(M + \frac{v}{V}Z\right),$$

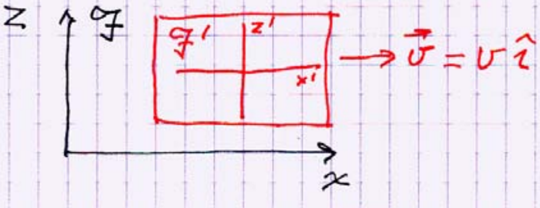
$$Z' = \beta\left(Z + \frac{v}{V}M\right), \qquad N' = \beta\left(N - \frac{v}{V}Y\right).$$

Electromagnetism and Relativity (I) 12.1/1

Review of Special Relativity

Galileo (17th century) supposed, in his own way, that the laws of physics are the same in all inertial frames.

- The Galilean transformation



$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Consider an event that occurs w.r.t. $\mathcal{F}$ at $(x, y, z, t) = (\alpha, \beta, \gamma, T)$;
the same event observed w.r.t. $\mathcal{F}'$ occurs at

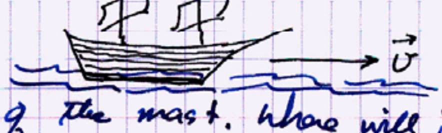
$$(x', y', z', t') = (\alpha - vT, \beta, \gamma, T).$$

Example Drop a ball from height H, in frame $\mathcal{F}'$.

$\mathcal{F}'$ coordinates : $(x', y', z', t') = (0, 0, H - \frac{1}{2}gT^2, T)$

$\mathcal{F}$ coordinates : $(x, y, z, t) = (vT, 0, H - \frac{1}{2}gt^2, T)$

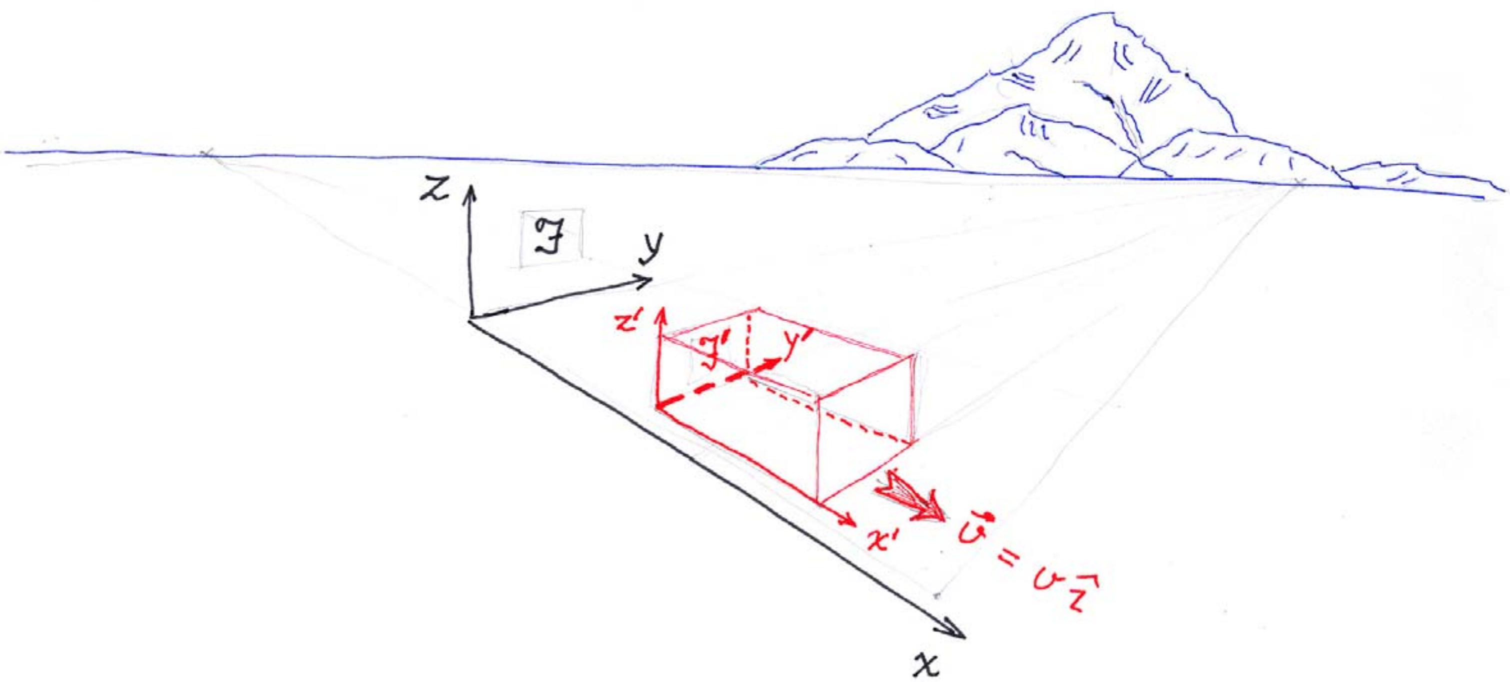
Galileo's Example

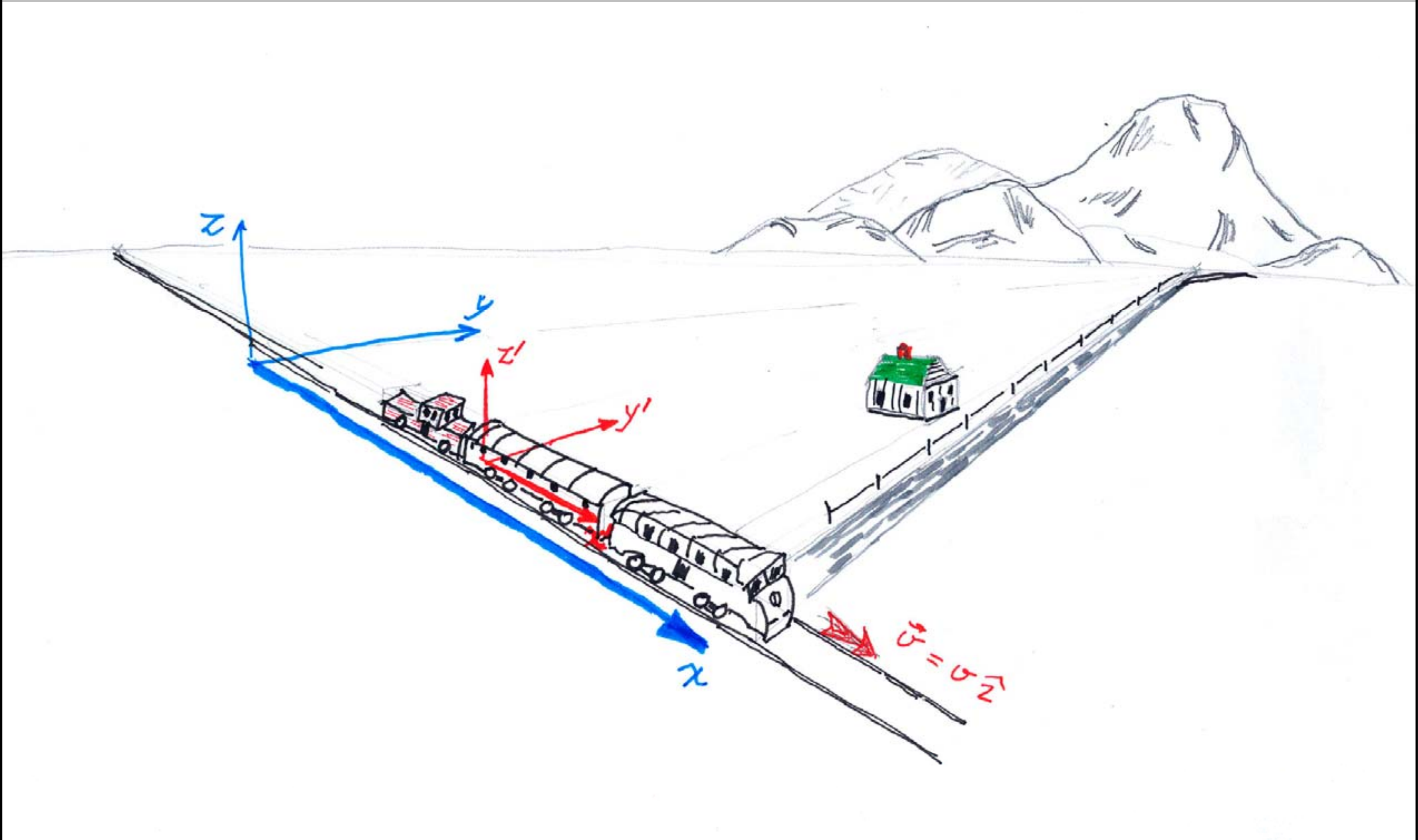


Drop a rock from the top of the mast. Where will it hit the deck?

Aristotelian physicists – Galileo's opponents – said that the rock would hit the deck in back of the mast, because the rock falls down while the mast moves forward.

... at the base of the mast.


z
$\mathcal{J}$
y
x
z'
$\mathcal{J}'$
y'
x'
$\vec{v} = v\hat{\imath}$


z
y
x
z'
y'
$\vec{v} = v\hat{z}$

Einstein (1905) 12.1/2

Start with two axioms:

(1) The laws of physics are the same in all inertial frames.

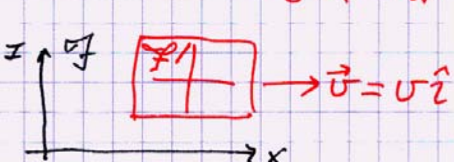
(2) The speed of light (in vacuum) is the same in all inertial frames.

These axioms, together, are consistent with the statement that Maxwell's equations describe electromagnetism in all inertial frames.

Then the coordinate transformations $\mathcal{F} \longleftrightarrow \mathcal{F}'$, the Lorentz transformations, are ...

$x' = \gamma(x - vt)$	$x = \gamma(x' + vt')$
$y' = y$	$y = y'$
$z' = z$	$z = z'$
$ct' = \gamma\left(ct - \frac{vx}{c}\right)$	$ct = \gamma\left(ct' + \frac{vx'}{c}\right)$
$\mathcal{F}' \leftarrow \mathcal{F}$	$\mathcal{F} \leftarrow \mathcal{F}'$

$$\boxed{\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}}$$



Consequences
- Lorentz contraction
- Time dilation
- Addition of velocities

Example Suppose 2 events occur at the origin of $\mathcal{F}'$, with time delay $T' = t_2' - t_1'$.

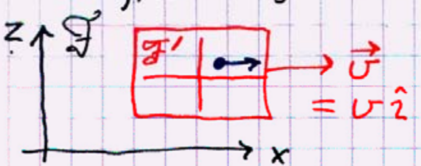
Then the time delay relative to the frame $\mathcal{F}$ is

$$t_2 - t_1 = \gamma \left[t_2' - t_1' + \frac{v}{c^2}(\underbrace{x_2'}_{=0} - \underbrace{x_1'}_{=0}) \right] \quad \text{(origin)}$$

$$t_2 - t_1 = \gamma T'$$

$$t_2 - t_1 = \frac{T'}{\sqrt{1 - v^2/c^2}}$$ i.e., <u>time dilation</u> by the factor γ

Example Suppose an object moves with constant velocity $\vec{w} = w\hat{\imath}$ in the frame $\mathcal{F}'$. Then its velocity relative to the coordinates of frame $\mathcal{F}$ is $\vec{u} = u\hat{\imath}$, where



$$u = \frac{\Delta x}{\Delta t} = \frac{\gamma(\Delta x' + v\Delta t')}{\gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)}$$

$$\boxed{u = \frac{w + v}{1 + \frac{vw}{c^2}}}$$

addition of velocities

Note: $\frac{\Delta x'}{\Delta t'} = w$

- If $v = 0.1c$ and $w = 0.1c$ then $u = \frac{0.1 + 0.1}{1 + 0.01}c = 0.198c$. (not 0.2 c !)
- If $v = 0.9c$ and $w = 0.9c$ then $u = \frac{0.9 + 0.9}{1 + 0.81}c = 0.995c$. (not 1.8 c !)
- If $w = c$ then $u = \frac{c + v}{1 + v/c} = c$.

A photon moves with the same speed (c) in either frame.

Relativistic particle dynamics

12.1/4

A particle (mass m) moves relative to $\mathcal{F}$ with velocity $\vec{v} = v\hat{x}$. (It is at rest in $\mathcal{F}'$.)

<u>Proper time</u> Define $d\tau$ by

$$c^2 (d\tau)^2 = c^2 (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$
$$= (c^2 - v^2)(dt)^2$$

$$d\tau = \frac{dt}{\gamma} \quad \text{because} \quad \boxed{\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}}$$

- $d\tau = dt'$ in the rest frame of the particle $(\gamma = 1)$
- $d\tau$ is Lorentz invariant (i.e., a "scalar")

$$(d\tau)^2 = \frac{(dt_1)^2}{\gamma_1^2} \quad \text{using coordinates of } \mathcal{F}_1$$
$$(d\tau)^2 = \frac{(dt_2)^2}{\gamma_2^2} \quad \text{using coordinates of } \mathcal{F}_2$$

$(d\tau)^2$ is the same. <u>Proof</u>

Transformation $\mathcal{F}_1 \to \mathcal{F}_2$

$$c\,dt_2 = \gamma_{21}\left[c\,dt_1 - \frac{v_{21}}{c} x_1\right]$$
$$dx_2 = \gamma_{21}\left[dx_1 - v_{21}\,dt_1\right]$$
$$dy_2 = dy_1 \quad \text{and} \quad dz_2 = dz_1$$

Let $X_{(2)} = c^2 (dt_2)^2 - (dx_2)^2 - (dy_2)^2 - (dz_2)^2$

$$= \gamma_{21}^2\left(c\,dt_1 - \frac{v_{21}}{c}dx_1\right)^2 - \gamma_{21}^2\left(dx_1 - v_{21}dt_1\right)^2 - (dy_1)^2 - (dz_1)^2$$

$$= \gamma_{21}^2\left(1 - \frac{v_{21}^2}{c^2}\right)c^2 dt_1^2 - \gamma_{21}^2\left(1 - \frac{v_{21}^2}{c^2}\right)(dx_1)^2 - (dy_1)^2 - (dz_1)^2$$
$$- 2\gamma_{21}^2 (c\,dt_1)\, v_{21}\frac{dx_1}{c} + 2\gamma_{21}^2 (v_{21}\,dt_1)\,dx_1$$

$$= c^2 (dt_1)^2 - (dx_1)^2 - (dy_1)^2 - (dz_1)^2 = X_{(1)}$$

$\therefore$ $d\tau$ is invariant. **QED**

Minkowski Space

12.1/5

$\hookrightarrow$ or, Spacetime

"Coordinate position" in Minkowski space is a 4-dimensional vector $\rightarrow$ locates an event

$$x^\mu = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \quad \text{a column vector}$$

x^μ = the position of an event in spacetime, relative to the coordinates of inertial frame $\mathcal{F}$.

($\mu = 0, 1, 2, 3$ and $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$)

The coordinates of the same event, relative to the coordinates of the frame $\mathcal{F}'$, are

$$x'^\mu = \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix}$$

$\beta = \frac{v}{c}$; $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

The Lorentz transformation is

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu \quad \text{where} \quad \Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Verify

$ct' = x'^0 = \gamma(ct) - \beta\gamma(x) = \gamma\left(ct - \frac{vx}{c}\right)$ ✓

$x' = x'^0 = -\beta\gamma(ct) + \gamma(x) = \gamma(x - vt)$ ✓

$y' = y$ and $z' = z$ ✓

$$\boxed{x'^\mu = \Lambda^\mu{}_\nu x^\nu}$$ Einstein summation convention

4 - momentum

Again, a particle (mass m) moves relative to $\mathcal{F}$ with velocity $\vec{v} = v\hat{\imath}$. (It is at rest in $\mathcal{F}'$!)

Define $p^\mu = m\dfrac{dx^\mu}{d\tau}$ $\quad (\mu = 0,1,2,3)$

where $d\tau$ is the proper time interval,

$$(d\tau)^2 = (dt')^2 = \underbrace{(dt)^2 - \frac{(dx)^2}{c^2} - \frac{(dy)^2}{c^2} - \frac{(dz)^2}{c^2}}$$
$$= \left(1 - \frac{v^2}{c^2}\right)(dt)^2$$

Energy and momentum

$$p^\mu = \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix}$$

The 4-momentum is also called the energy-momentum four-vector.

Just as time and space make a coordinate four-vector, energy and momentum make a four-vector

Take $\vec{v} = v\hat{\imath}$.

- $p_x = m\dfrac{dx}{d\tau} = \dfrac{m\,dx}{dt/\gamma} = mv\gamma$
- $p_y = 0$ and $p_z = 0$ for $\vec{v} = v\hat{\imath}$.
- $\dfrac{E}{c} = m\dfrac{dx^0}{d\tau} = \dfrac{m\,c\,dt}{dt/\gamma} = mc\gamma$

Quiz question

The rest mass of a muon is 106 MeV/c^2. The mean lifetime, in the rest frame, is 2.2 ms.

(A) Determine the mean lifetime of a muon with energy 424 MeV.

(B) Determine would far it would travel during one mean lifetime.