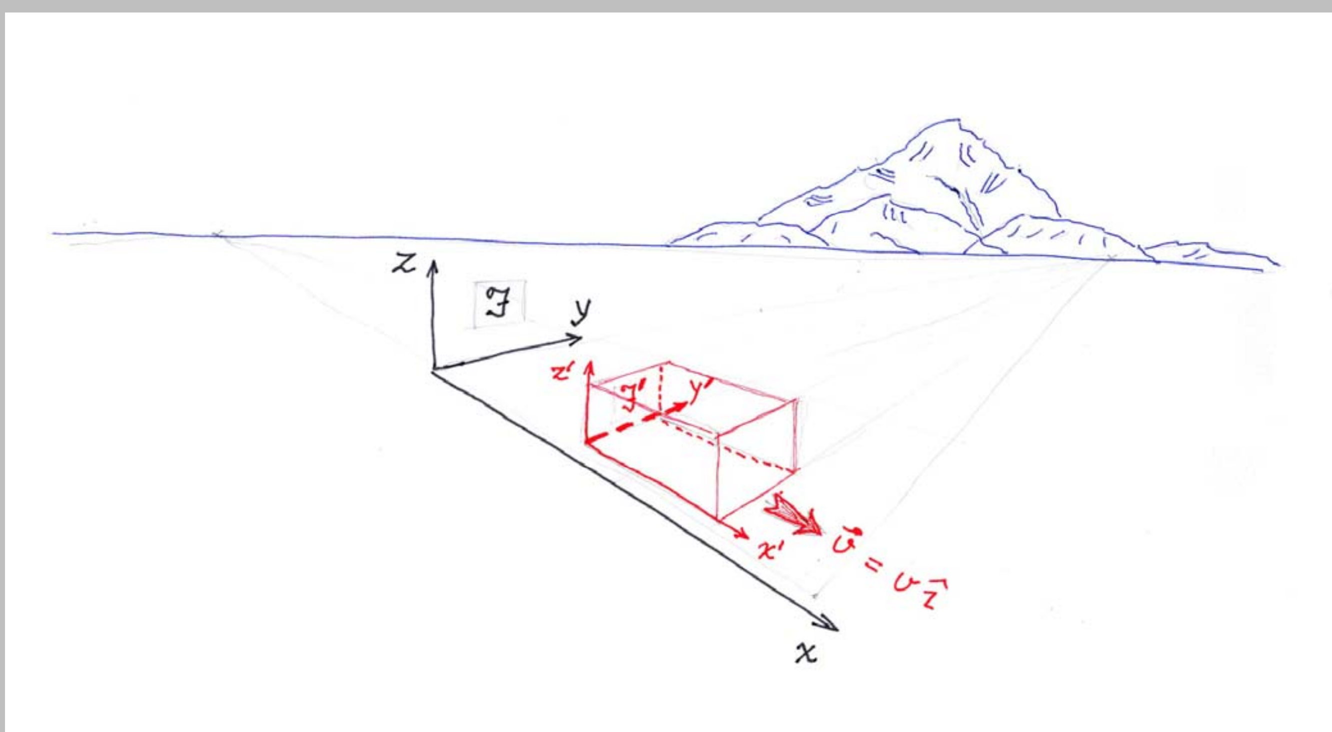
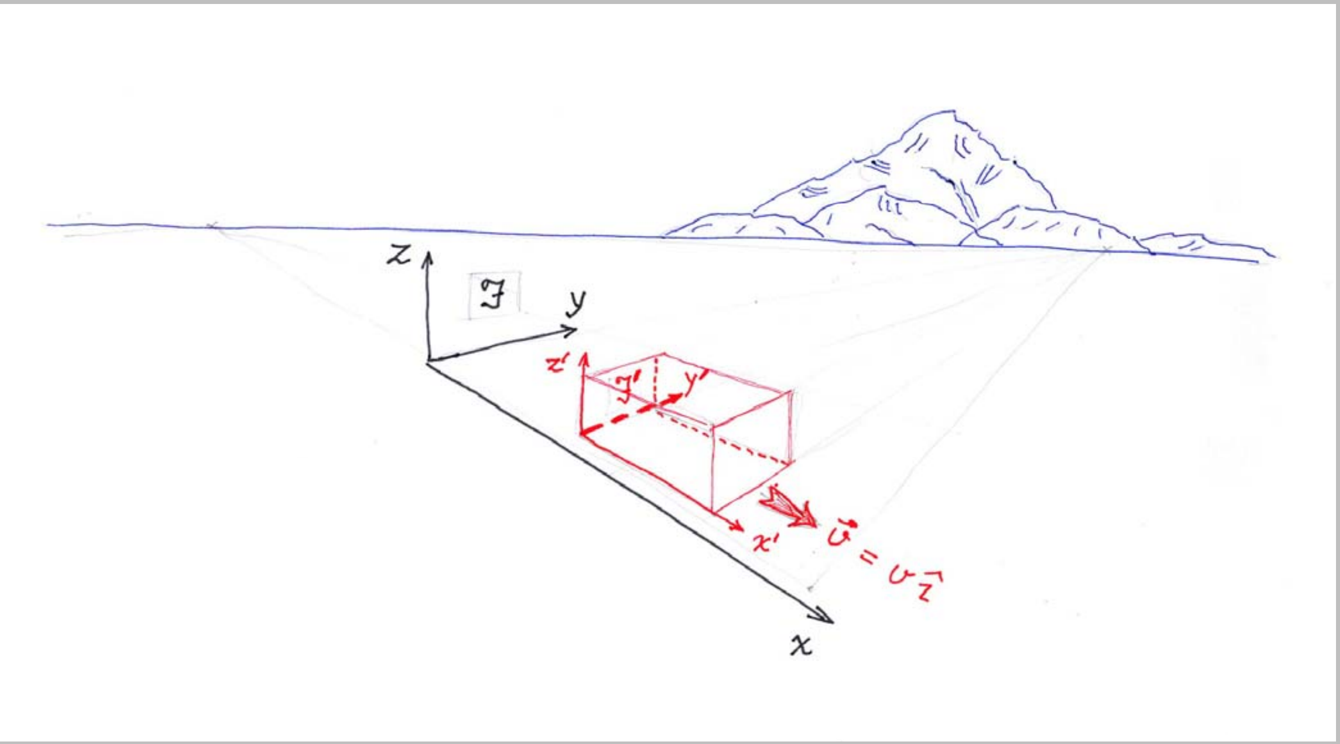
Electromagnetism and Relativity (II) 12.2/1

The Principle of Relativity : the laws of physics are the same in all <u>inertial frames</u>

<u>Inertial frame</u> — a frame of reference in which the "law of inertia" is true ; i.e., an object in motion remains in motion <u>with constant velocity</u> if no force is acting on it. The opposite of an inertial frame is an accelerating frame.



Frame $\mathcal{F}'$ moves with velocity $\vec{v} = v\hat{x}$ relative to frame $\mathcal{F}$.



The Lorentz transformation

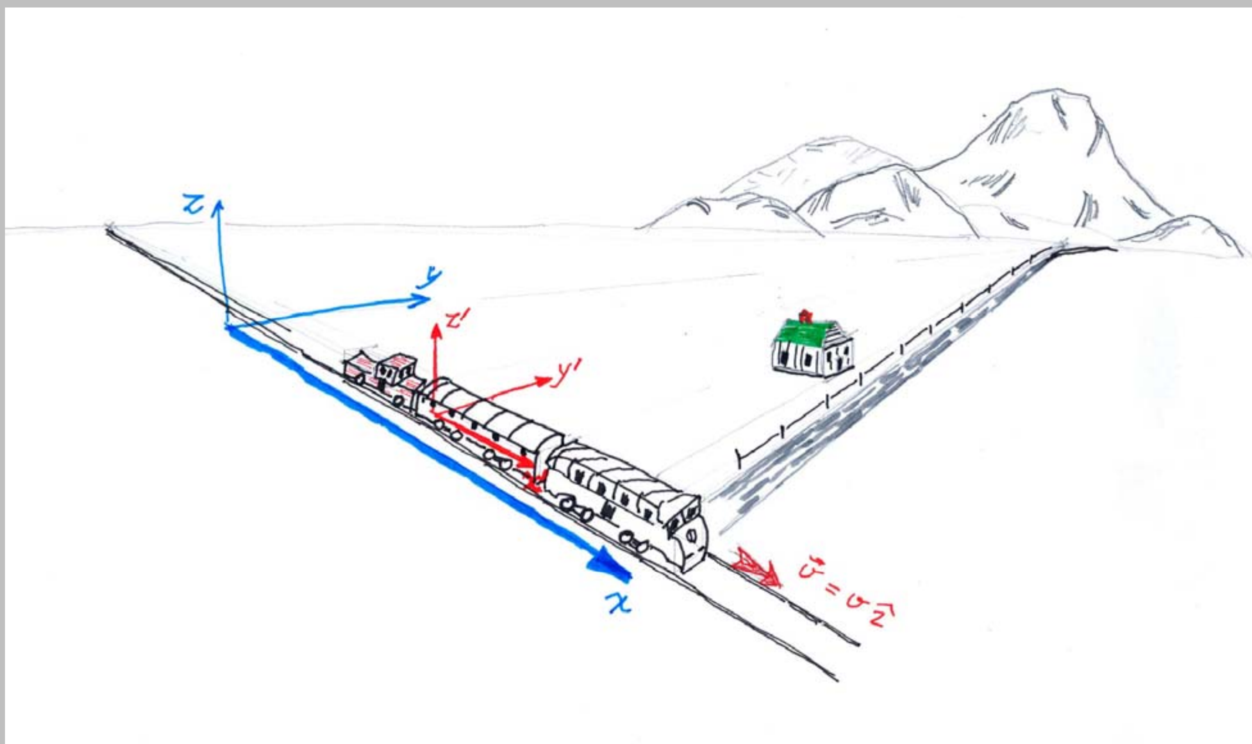
$$ct' = \gamma\left(ct - \frac{v}{c}x\right) \qquad ct = \gamma\left(ct' + \frac{v}{c}x'\right)$$

$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$

$$y' = y \qquad y = y'$$

$$z' = z \qquad z = z'$$

$$\mathcal{F}' \leftarrow \mathcal{F} \qquad \mathcal{F} \leftarrow \mathcal{F}'$$



How shall we write the equations of physics, to guarantee that they have the same form (and make the same predictions) in all inertial frames?

<u>THEOREM 1</u> An equation is "manifestly covariant" if it is written in <u>tensor form</u>.

Minkowski space and Tensor analysis 12.2/2

— vectors, scalars, and tensors

The position of an event in spacetime

$$x^\mu = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$ in coordinates of an inertial frame $\mathcal{F}$.

The same event, observed relative to the inertial frame $\mathcal{F}'$ has the coordinates

$$x'^\mu = \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix}$$

$\boxed{\beta = v/c \quad \gamma = 1/\sqrt{1-v^2/c^2}}$

The transformation

$$\underset{\mathcal{F}'}{x'^\mu} = \Lambda^\mu{}_\nu \underset{\mathcal{F}}{x^\nu} \qquad \mathcal{F}' \leftarrow \mathcal{F}$$

where

$$\Lambda^\mu{}_\nu = \begin{array}{c|cccc} \mu \backslash \nu & 0 & 1 & 2 & 3 \\ \hline 0 & \gamma & -\beta\gamma & 0 & 0 \\ 1 & -\beta\gamma & \gamma & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

— the Lorentz transformation matrix —

E.g. $x'^0 = \gamma x^0 - \beta\gamma x^1$

$ct' = \gamma(ct - \frac{v}{c}x)$ etc

Define "Vector": A vector is a 4-component quantity V^μ ($\mu = 0,1,2,3$) that transforms in the same way as x^μ; i.e.,

$$\underset{\mathcal{F}'}{V'^\mu} = \Lambda^\mu{}_\nu \underset{\mathcal{F}}{V^\nu} \qquad \mathcal{F}' \leftarrow \mathcal{F}$$

Definition of vector
$V'^\mu = \Lambda^\mu{}_\nu V^\nu$

Vectors, i.e., spacetime vectors ... 12.2/3

$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$ $\quad\quad$ $\Lambda^{\mu}{}_{\nu}$ = Lorentz transformation matrix

$V'^{\mu} = \Lambda^{\mu}{}_{\nu}\, V^{\nu}$ $\quad \longleftarrow$ definition of a vector

Examples

- x^{μ}
- $3x^{\mu}$
- $a x^{\mu}$ for any scalar a
- $x_1^{\mu} \pm x_2^{\mu}$
- any linear combinations of vectors
- We'll see other examples later, like 4-velocity and 4-momentum

Define "Scalar"

A scalar is a quantity S that has the same value in all inertial frames.

I.e., $\quad S'(x') = S(x) \quad \longleftarrow$ where $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$

THEOREM 2 Let V^μ and W^μ be Lorentz vectors. Then $V \cdot W$, defined by $g_{\mu\nu} V^\mu W^\nu$, is a scalar.

The metric tensor

$$g_{\mu\nu} = \begin{array}{c|cccc} \mu \backslash \nu & 0 & 1 & 2 & 3 \\ \hline 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

PROOF of theorem 2

$$V' \cdot W' = g_{\mu\nu} V'^\mu W'^\nu \quad \text{(Einstein summation convention!)}$$

$$= g_{\mu\nu} \Lambda^\mu{}_\rho V^\rho \Lambda^\nu{}_\sigma W^\sigma \quad \text{(Einstein summation convention!)}$$

↑ everywhere!!

$$= V^\rho \{ \underbrace{\Lambda_\rho{}^\mu g_{\mu\nu} \Lambda^\nu{}_\sigma} \} W^\sigma$$

a product of 3 matrices, call it $X_{\rho\sigma}$.
Calculate it below ☆.

Thus $V' \cdot W' = g_{\rho\sigma} V^\rho W^\sigma = V \cdot W$. Q.E.D.

$V \cdot W$ has the same value in all inertial frames ⇒ a scalar.

☆
$$X_{\rho\sigma} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\begin{bmatrix} -\gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$X_{\rho\sigma} = \begin{bmatrix} -\gamma^2 + \beta^2\gamma^2 & \beta\gamma^2 - \beta\gamma^2 & 0 & 0 \\ \beta\gamma^2 - \beta\gamma^2 & -\beta^2\gamma^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = g_{\rho\sigma}$$

$$\boxed{\gamma^2(1-\beta^2) = 1}$$

Metric Tensor and Scalar Product 12.2/5

$g_{\mu\nu}$	$\mu\nu$
-1	00
$+1$	11, 22, 33
0	$\mu \neq \nu$

$$a \cdot b = g_{\mu\nu} a^\mu b^\nu \quad (\textstyle\sum_{\mu,\nu} \text{ implied!})$$

$$a \cdot b = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

$$a \cdot b = -a^0 b^0 + \vec{a} \cdot \vec{b}$$

Or, diag $(-1, 1, 1, 1)$

{Notation: a^μ is a 4-vector } spacetime
$\vec{a}$ is a 3-vector } space only

⊕ Examples

$$x^2 = x \cdot x = -(ct)^2 + x^2 + y^2 + z^2$$

$$x'^2 = x' \cdot x' = -(ct')^2 + x'^2 + y'^2 + z'^2$$

$x^2 = x'^2$ by Theorem 2. Therefore the speed of light is constant. The "light cone":

$$x^2 = 0 \implies ct = \sqrt{x^2 + y^2 + z^2}$$

$$x'^2 = 0 \implies ct' = \sqrt{x'^2 + y'^2 + z'^2}$$

These are the same points in 2 reference frames; c is the same.

⊕ Proper time (τ) of an object in space time

$$c^2 (d\tau)^2 = c^2 (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

REMEMBER? From Last Time?

$$c^2 (d\tau)^2 = -g_{\mu\nu} dx^\mu dx^\nu$$

$$= -(dx) \cdot (dx);$$

SO Proper Time is a scalar by Theorem 2

$$dx \cdot dx = dx' \cdot dx'$$

← special case of the general result $V \cdot W = V' \cdot W'$

⊕ Vectors used in particle dynamics

Define 4-velocity by $\eta^\mu = \dfrac{dx^\mu}{d\tau}$

Define 4-momentum by $p^\mu = m\eta^\mu = m \dfrac{dx^\mu}{d\tau}$

Do you see why these quantities are vectors?

REMEMBER? From Last Time?

Tensor Analysis

12.2/6

A tensor $T^{\mu\nu}$ is a quantity that transforms in the same way as $x^\mu x^\nu$.

{I.e., this is a tensor with rank = 2}

$$T'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma}$$

$$\mathcal{F}' \longleftarrow \mathcal{F}$$

Theorem 1 An equation is "manifestly covariant" if it is written in tensor form.

$$\boxed{T_1^{\mu\nu} = T_2^{\mu\nu}}$$

Similar idea in classical mechanics:

Newton's second law $\vec{F} = m\vec{a}$.

That's "tensor form" for ordinary 3D vectors.

i.e. 3D vectors

Newton's second law is "manifestly covariant" with respect to 3D rotations. $\vec{F}$ and $m\vec{a}$ are both vectors.

Proof of the theorem Suppose $T_1^{\mu\nu} = T_2^{\mu\nu}$ (frame $\mathcal{F}$)

Now consider

$$T_1'^{\mu\nu} - T_2'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T_1^{\rho\sigma} - \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T_2^{\rho\sigma}$$

$$= \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \left[\underbrace{T_1^{\rho\sigma} - T_2^{\rho\sigma}}_{=0} \right]$$

$$= 0$$

So $T_1'^{\mu\nu} = T_2'^{\mu\nu}$. Q.E.D.

12.2/7

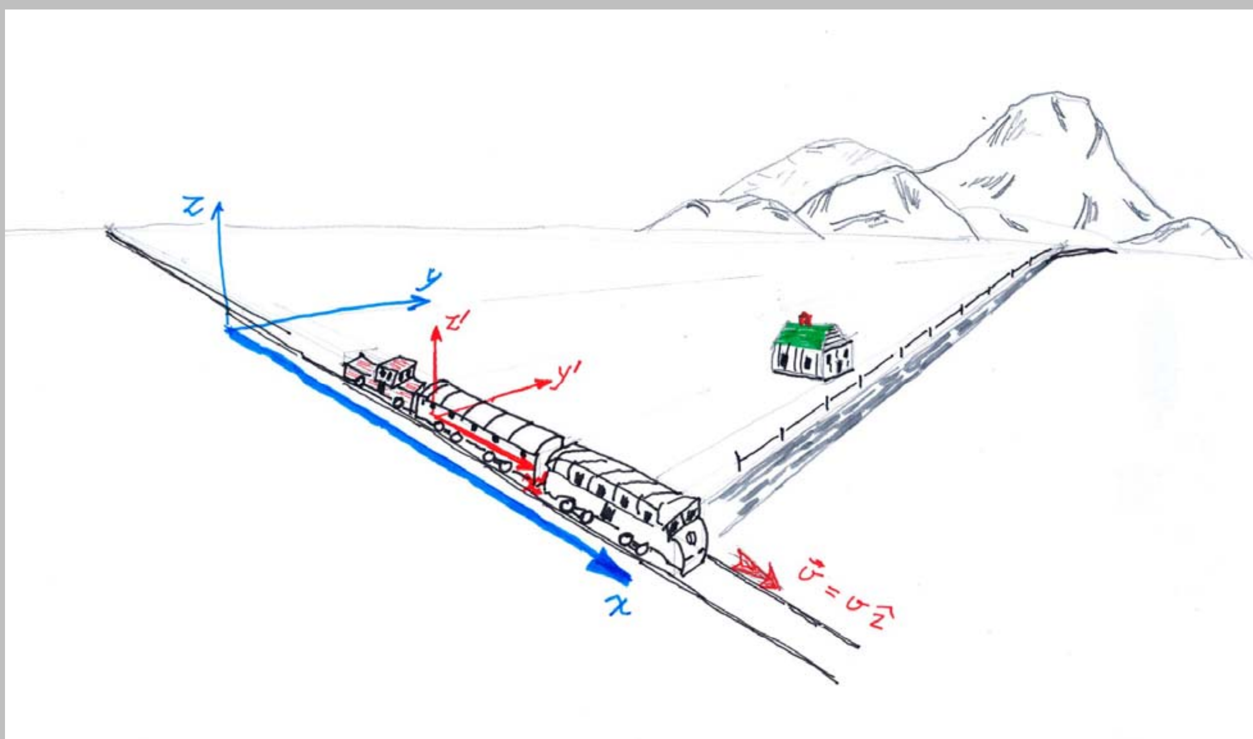
So here is what we'll do after the spring break:

Write the equations of electromagnetism in <u>tensor form</u>

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \mu_0 J^{\mu} \quad \text{and} \quad \frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0 ,$$

$$\frac{dp^{\mu}}{d\tau} = K^{\mu} = q F^{\mu\nu} \eta_{\nu} .$$

That will tell us how the fields transform between $\mathcal{F}$ and $\bar{\mathcal{F}}$!



Quiz Question:
Where are you going for Spring Break?