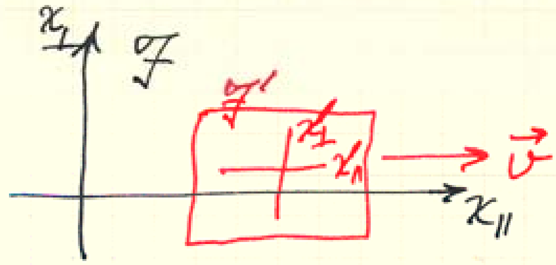
# Three Examples

(1) The Quiz Question from last time.




Suppose there is an electrostatic system w.r.t. inertial frame $\mathcal{F}'$:

$$\nabla' \cdot \vec{E}' = \frac{\rho'}{\epsilon_0}, \quad \nabla' \times \vec{E}' = 0, \quad \vec{B}' = 0.$$

Frame $\mathcal{F}'$ moves with velocity $\vec{v}$ relative to another inertial frame $\mathcal{F}$. Then the fields observed in $\mathcal{F}$ are

$$E_\parallel = E'_\parallel \qquad B_\parallel = 0$$
$$\vec{E}_\perp = \gamma \vec{E}'_\perp \qquad \vec{B}_\perp = \frac{\gamma}{c^2} \vec{v} \times \vec{E}'_\perp$$

(see Table 12.3).

## Calculate $\nabla \times \vec{B}$

First, note that $\vec{B} = \frac{1}{c^2} \vec{v} \times \vec{E}$.

Proof $B_\parallel = 0$ so $\vec{B} = \vec{B}_\perp = \frac{\gamma}{c^2} \vec{v} \times \vec{E}'_\perp$

$$= \frac{\vec{v}}{c^2} \times \vec{E}_\perp = \frac{\vec{v}}{c^2} \times \vec{E}$$ because $\vec{v} \times \vec{E}_\parallel = 0$

Now, $\nabla \times \vec{B} = \nabla \times \left( \frac{\vec{v}}{c^2} \times \vec{E} \right)$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\, \vec{A} \cdot \vec{C} - \vec{C}\, \vec{A} \cdot \vec{B}$$

$$\nabla \times \vec{B} = \frac{\vec{v}}{c^2} \nabla \cdot \vec{E} - \left( \frac{\vec{v}}{c^2} \cdot \nabla \right) \vec{E}$$

$$= \frac{\vec{v}}{c^2} \frac{\rho}{\epsilon_0} - \frac{v}{c^2} \frac{\partial \vec{E}}{\partial x_\parallel}$$

$\vec{E}$ depends on $x_{\parallel} - vt$.

$$\vec{E}(x_{\parallel}, \vec{x}_{\perp}, t) = E'_{\parallel}(x'_{\parallel}, \vec{x}'_{\perp})\,\hat{\varepsilon} + \gamma \vec{E}'_{\perp}(x'_{\parallel}, \vec{x}'_{\perp})$$
$$= E'_{\parallel}(\underbrace{\gamma(x_{\parallel}-vt)}_{x_{\parallel}-vt}, \vec{x}_{\perp}) + \gamma \vec{E}'_{\perp}(\underbrace{\gamma(x_{\parallel}-vt)}_{x_{\parallel}-vt}, \vec{x}_{\perp})$$

Thus $\dfrac{\partial \vec{E}}{\partial t} = -v \dfrac{\partial \vec{E}}{\partial x_{\parallel}}$

So $$\nabla \times \vec{B} = \frac{\vec{v}}{c^2}\frac{\rho}{\epsilon_0} - \frac{v}{c^2}\frac{(-1)}{v}\frac{\partial \vec{E}}{\partial t}$$

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{where } \vec{J} = \rho \vec{v}}$$

That's right — it's the Ampere-Maxwell equation!

Check units: $\vec{J} = \rho \vec{v}$

$$\frac{A}{m^2} = \frac{C}{m^3}\frac{m}{s}$$

- By the way, what is $\rho$?

$$\rho(\vec{x}, t) = \epsilon_0 \nabla \cdot \vec{E} = \epsilon_0 \left\{ \frac{\partial E_{\parallel}}{\partial x_{\parallel}} + \frac{\partial \vec{E}_{\perp}}{\partial \vec{x}_{\perp}} \right\}$$
$$= \epsilon_0 \left\{ \frac{\partial E'_{\parallel}}{\partial x_{\parallel}} + \frac{\gamma\, \partial \vec{E}'_{\perp}}{\partial \vec{x}'_{\perp}} \right\}$$
$$= \epsilon_0 \gamma \left\{ \frac{1}{\gamma}\frac{\partial E'_{\parallel}}{\partial x_{\parallel}} + \underbrace{\nabla' \cdot \vec{E}'}_{\rho'/\epsilon_0} - \frac{\partial E'_{\parallel}}{\partial x'_{\parallel}} \right\}$$

these cancel

$x'_{\parallel} = \gamma(x_{\parallel} - vt)$

$\delta x'_{\parallel} = \gamma\, \delta x_{\parallel}$ with $t$ fixed

$$= \gamma \rho'(\vec{x}')$$ (alternate proof?)

$$\rho(x_{\parallel}, \vec{x}_{\perp}, t) = \gamma\ \rho'(\gamma(x_{\parallel} - vt), \vec{x}_{\perp})$$

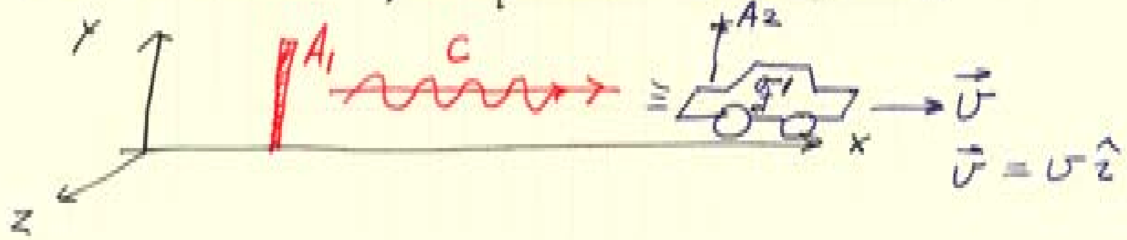
position moves with velocity $\vec{v} = v\hat{\varepsilon}$

density is increased because of Lorentz contraction

$\rho$ is not a Lorentz scalar!

Exercise: Prove div B =

(2) An earlier quiz question : 2 antennas




Let $S$ be the rest frame of $A_1$, the transmitter.

Then $\vec{E}(\vec{x},t) = E_0 \hat{j} \cos(kx-\omega t)$

$\vec{B}(\vec{x},t) = \frac{E_0}{c} \hat{k} \cos(kx-\omega t)$

Let $S'$ be the rest frame of $A_2$, the receiver.

$E'_x = E_x = 0$ USING TABLE 12.2

$E'_y = \gamma(E_y - vB_z) = \gamma\left(1-\frac{v}{c}\right) E_0 \cos(kx-\omega t)$

$E'_z = \gamma(E_z + vB_y) = 0$

$B'_x = B_x = 0$

$B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) = 0$

$B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right) = \gamma\left(1-\frac{v}{c}\right) \frac{E_0}{c} \cos(kx-\omega t)$

Also

$$
\begin{aligned}
kx - \omega t &= k\gamma(x' + vt') - \omega\gamma\left(t' + \frac{v}{c^2}x'\right) \\
&= \gamma\left(k - \frac{v}{c^2}\omega\right)x' - \gamma(\omega - vk)t' \\
&= k'x' - \omega' t'
\end{aligned}
$$

where $k' = \gamma\left(k - \frac{v}{c^2}\omega\right)$ and $\omega' = \gamma(\omega - vk)$.

Note that $\omega = ck$ so that $c$ is the phase velocity $\frac{\omega}{k}$.

Then also $\omega' = ck'$ so the speed of light is the same in both reference frames.

$\omega' = \gamma\left(\omega - \frac{v}{c}\omega\right) = \gamma\left(1-\frac{v}{c}\right)\omega$

$k' = \gamma\left(k - \frac{v}{c}k\right) = \gamma\left(1-\frac{v}{c}\right)k$

$\omega'/k' = \omega/k = c$

## What is the frequency for antenna $A_2$?

$k = \omega/c$

$$f' = \frac{\omega'}{2\pi} = \frac{\gamma(\omega - vk)}{2\pi}$$

$$f' = \gamma\left(1 - \frac{v}{c}\right) f$$

$$f' = \sqrt{\frac{1 - v/c}{1 + v/c}}\, f$$ ← Doppler shift for light

[The Doppler shift for sound is $f' = \left(1 - \frac{v}{c_s}\right) f$.]

## What is the amplitude of oscillation of the electric field?

$$\vec{E}'(\vec{x}', t') = \sqrt{\frac{1 - v/c}{1 + v/c}}\, E_0 \cos(k'x' - \omega' t')\, \hat{j}$$

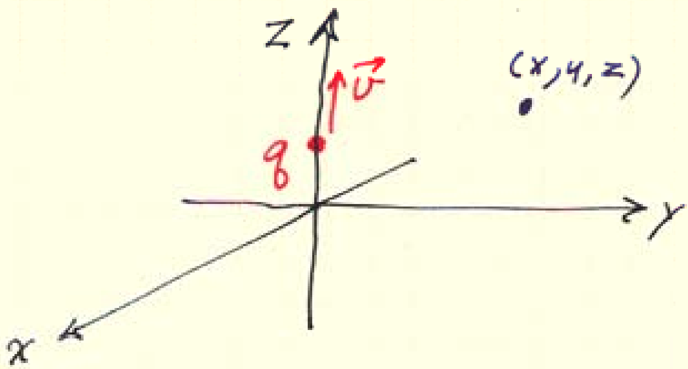
$$\vec{B}'(\vec{x}', t') = \sqrt{\frac{1 - v/c}{1 + v/c}}\, \frac{E_0}{c} \cos(k'x' - \omega' t')\, \hat{k}$$

$$\therefore\ E_0' = \sqrt{\frac{1 - v/c}{1 + v/c}}\, E_0$$

- The Lorentz transformation of a polarized plane wave is a polarized plane wave.
- As $v \to c$, $f' \to 0$ and $E_0' \to 0$.

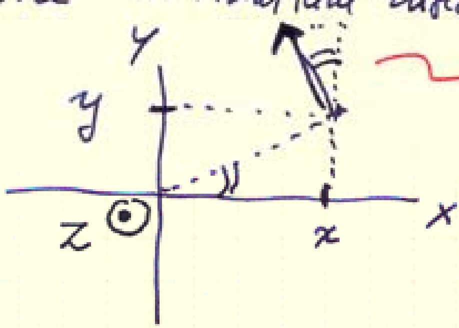
  I.e., the light gets redder and dimmer.

(3) From the magnetic field due to a moving charged particle, derive the magnetic field due to a line of current




$$\vec{B}(\vec{x},t) = \frac{\mu_0}{4\pi}\, q\gamma v\, \frac{-y\,\hat{\imath} + x\,\hat{\jmath}}{\left[x^2+y^2+\gamma^2(z-vt)^2\right]^{3/2}}$$

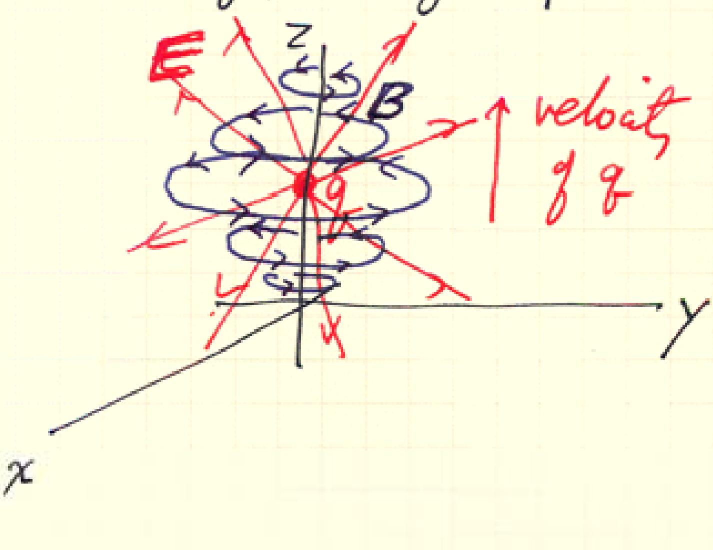
The azimuthal direction around the z axis:



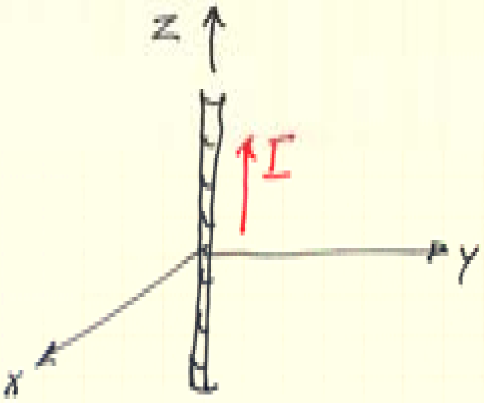

— the magnetic field is azimuthal

$$\hat{\phi} = \frac{-y\hat{\imath} + x\hat{\jmath}}{\sqrt{x^2+y^2}}$$

A single charge produces a pulse of $\vec{E}$ and $\vec{B}$

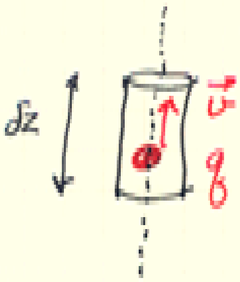



Now calculate the magnetic field due to a linear current $I\hat{k}$, by integration.




A long straight wire with current $I$.

Subdivide the z-axis into small segments $\delta z$.




Imagine that in $\delta z$ a single charge $q$ moves with velocity $v\hat{z}$. The time for $q$ to pass through $\delta z$ is $\delta t = \frac{\delta z}{v}$.

Then the current $= \frac{Q}{\text{Time}} = \frac{q}{\delta t} = \frac{qv}{\delta z}$

$\therefore \quad I\,dz = qv$

Check units : $A \cdot m = C \cdot \frac{m}{s}$ ✓

Use cylindrical coordinates $(r, \phi, z)$ $\qquad x = r\cos\phi$, $y = r\sin\phi$

$$d\vec{B} = \frac{\mu_0}{4\pi} \underbrace{qv}_{qv \,=\, I\,dz} \gamma \frac{r\hat{\phi}}{\left[r^2 + \gamma^2(z-vt)^2\right]^{3/2}}$$

So, integrating along the wire, $\vec{B} = B_\phi \hat{\phi}$

$$B_\phi = \frac{\mu_0}{4\pi} I \int_{-\infty}^{\infty} \frac{r\gamma\,dz}{\left[r^2 + \gamma^2(z-vt)^2\right]^{3/2}}$$

$$B_\phi = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{r \gamma \, dz}{\left[ r^2 + \gamma^2 (z - vt)^2 \right]^{3/2}}$$

Change the variable of integration from $z$ to $\zeta$ where

$$r \zeta = \gamma (z - vt)$$

$$r \, d\zeta = \gamma \, dz$$

$$B_\phi = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{r^2 \, d\zeta}{\left[ r^2 + r^2 \zeta^2 \right]^{3/2}}$$

$$= \frac{\mu_0 I}{4\pi} \frac{r^2}{r^3} \underbrace{\int_{-\infty}^{\infty} \frac{d\zeta}{(1+\zeta^2)^{3/2}}}_{2}$$

The antiderivative is $\zeta / \sqrt{1+\zeta^2}$

$$B_\phi = \frac{\mu_0 I}{2\pi r}$$

[redacted]