Radiation of Electromagnetic Waves R1/1

- The field equations — relating charge, current and fields.

$$\nabla\cdot\vec{E} = \rho/\epsilon_0 \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \nabla\times\vec{B} = \mu_0\left(\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

- The potential functions — simplify the solution of these equations

$$\vec{B} = \nabla\times\vec{A} \quad \text{and} \quad \vec{E} = -\frac{\partial\vec{A}}{\partial t} - \nabla V.$$

- Also, impose the Lorentz gauge condition

$$\nabla\cdot\vec{A} = \frac{-1}{c^2}\frac{\partial V}{\partial t} \qquad \text{(gauge fixing)}$$

Then $V(\vec{x},t)$ and $\vec{A}(\vec{x},t)$ obey these equations

$$-\nabla^2 V + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \rho/\epsilon_0$$

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \mu_0\vec{J}$$

The d'Alembertian

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \quad \text{a differential operator}$$

$$\Box V = \rho/\epsilon_0 \quad \text{and} \quad \Box\vec{A} = \mu_0\vec{J}$$

We can solve for V and $\vec{A}$ (in terms of ρ and $\vec{J}$) if we know the Green's function of $\Box$.

Green's Functions

Suppose we have a linear differential equation:

$$\mathcal{D} f = \sigma$$

$\mathcal{D}$: a differential operator

$f(x)$: the function we need to determine.
x could be multi-dimensional; e.g., $x = (\vec{x}, t)$

$\sigma(x)$: a known function; the source of f.

The Green's function $G(x; x')$ is defined by

$$\mathcal{D} G = \delta^n(x - x') ;$$

i.e., G is the function for a point source at x!

Then

$$f(x) = \int G(x; x')\, \sigma(x')\, d^n x'$$

The problem is solved; it is reduced to integration.

Proof $\quad \mathcal{D} f = \int \underbrace{\mathcal{D} G(x; x')}_{\delta^n(x-x')}\, \sigma(x')\, d^n x' = \sigma(x)$

Q.E.D.

Example 1: Electrostatics

$$-\nabla^2 V = \rho/\epsilon_0$$

$$-\nabla^2 \frac{1}{4\pi|\vec{x} - \vec{x}'|} = \delta^3(\vec{x} - \vec{x}')$$

$$\boxed{\text{The Green's function of } -\nabla^2 \text{ is } \frac{1}{4\pi|\vec{x} - \vec{x}'|}.}$$

$$V(\vec{x}) = \frac{1}{\epsilon_0} \int \frac{\rho(\vec{x}')}{4\pi|\vec{x} - \vec{x}'|}\, d^3x' = \int \frac{dQ}{4\pi\epsilon_0 r}$$

"reduced to integration"

Example 2: harmonic time dependence $\qquad$ P1/3

Suppose $\rho(\vec{x},t) = \tilde{\rho}(\vec{x})\, e^{-i\omega t}$ $\qquad$ /Real Part/

then $V(\vec{x},t) = \tilde{V}(\vec{x})\, e^{-i\omega t}$ $\qquad$ $/\ \frac{\partial}{\partial t} = -i\omega\ /$

w/ $\quad -\nabla^2 \tilde{V} - \frac{\omega^2}{c^2}\tilde{V} = \frac{\tilde{\rho}}{\epsilon_0}$

Theorem The Green's function of $-(\nabla^2 + k^2)$ is

$$\frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} \equiv \tilde{G}(\vec{x},\vec{x}').$$

Proof. We need to prove $\quad -\nabla^2\tilde{G} - k^2\tilde{G} = \delta^3(\vec{x}-\vec{x}')$

W.L.O.G., let $\vec{x}' = 0$.

- $$\nabla^2 \frac{e^{ikr}}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)\frac{e^{ikr}}{r}$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left\{ ikr\, e^{ikr} - e^{ikr} \right\} = \frac{1}{r^2}\frac{\partial}{\partial r}\left\{ (ikr-1)\, e^{ikr} \right\}$$
$$= \left[\frac{ik}{r^2} + \frac{ik}{r^2}(ikr-1) \right] e^{ikr} = -k^2 \frac{e^{ikr}}{r}$$

$$(\nabla^2 + k^2)\frac{e^{ikr}}{r} = 0 \qquad (\text{for } r \neq 0)$$

- Now consider $\int_V (\nabla^2 + k^2)\frac{e^{ikr}}{r}$

↙ Gauss's theorem

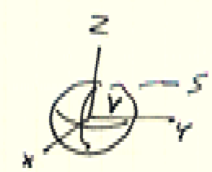
$$= \oint_S \hat{r}\cdot\nabla\left(\frac{e^{ikr}}{r}\right) dA + k^2 \int_V \frac{e^{ikr}}{r}\, d^3x$$

$$\xrightarrow[R\to 0]{} \frac{-1}{R^2}\cdot 4\pi R^2 + k^2 \cdot 2\pi R^2$$

$$\xrightarrow[R\to 0]{} -4\pi$$

$$\therefore\ (\nabla^2 + k^2)\frac{e^{ikr}}{r} = -4\pi\,\delta^3(\vec{x})$$

Q.E.D.



sphere of radius R; $R \to 0$

$\frac{4\pi r^2 dr}{r} = 4\pi r\, dr$

integral $= 2\pi R^2$

Result

If $\rho(\vec{x},t) = \tilde{\rho}(\vec{x})\, e^{-i\omega t}$ then

$$V(\vec{x},t) = \frac{1}{\epsilon_0}\int \frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|}\,\tilde{\rho}(\vec{x}')\, d^3x' \quad e^{-i\omega t} \qquad (*)$$

where $k = \frac{\omega}{c}$!

General time dependence

Arbitrary $\rho(\vec{x},t)$.

We can always write $\rho(\vec{x},t)$ as a Fourier integral.

$$\rho(\vec{x},t) = \int_{-\infty}^{\infty} \tilde{\rho}(\vec{x},\omega)\, e^{-i\omega t}\, d\omega$$

The equation $\Box V = \rho/\epsilon_0$ is linear in V.

The superposition principle holds. So

$$V(\vec{x},t) = \int_{-\infty}^{\infty} \tilde{V}(\vec{x},\omega)\, e^{-i\omega t}\, d\omega$$

where $\tilde{V}(\vec{x},\omega)$ is given by $(*)$.

$$V(\vec{x},t) = \frac{1}{\epsilon_0}\int_{-\infty}^{\infty}\int \frac{e^{i\omega|\vec{x}-\vec{x}'|/c}}{4\pi|\vec{x}-\vec{x}'|}\,\tilde{\rho}(\vec{x}',\omega)\, d^3x'\, e^{-i\omega t}\, d\omega$$

$$= \frac{1}{\epsilon_0}\int \frac{\rho(\vec{x}',\, t-|\vec{x}-\vec{x}'|/c)}{4\pi|\vec{x}-\vec{x}'|}\, d^3x'$$

Do you get this?

$$\int_{-\infty}^{\infty} \tilde{\rho}(\vec{x}',\omega)\, \underbrace{e^{i\omega|\vec{x}-\vec{x}'|/c}\, e^{-i\omega t}}_{e^{-i\omega t'}}\, d\omega = \rho(\vec{x}',t')$$

where $t' = t - \frac{|\vec{x}-\vec{x}'|}{c}$

Result for general time dependence R1/5

$$V(\vec{x},t) = \frac{1}{\epsilon_0}\int \frac{\rho(\vec{x}', t - |\vec{x}-\vec{x}'|/c)}{4\pi |\vec{x}-\vec{x}'|}\, d^3x'$$

& similarly

$$\vec{A}(\vec{x},t) = \mu_0 \int \frac{\vec{J}(\vec{x}', t - |\vec{x}-\vec{x}'|/c)}{4\pi |\vec{x}-\vec{x}'|}\, d^3x'$$

which are called the "retarded potentials" because they depend on the sources ($\rho(x')$ and $\vec{J}(x')$) at an earlier time $t' = t - \frac{|\vec{x}-\vec{x}'|}{c}$.

The "retardation" is $|\vec{x}-\vec{x}'|/c$ = the time for light to travel from $\vec{x}'$ (source point) to $\vec{x}$ (field point).

The Green's function of $\Box$

$$\Box G = \delta^3(\vec{x}-\vec{x}')\,\delta(t-t')$$

i.e., G is a time-dependent Green's function.

$G = G(\vec{x},t\,;\,\vec{x}',t')$

Theorem $\quad G = \dfrac{\delta(\tau - r/c)}{4\pi r}$ where $r = |\vec{x}-\vec{x}'|$, $\tau = t - t'$

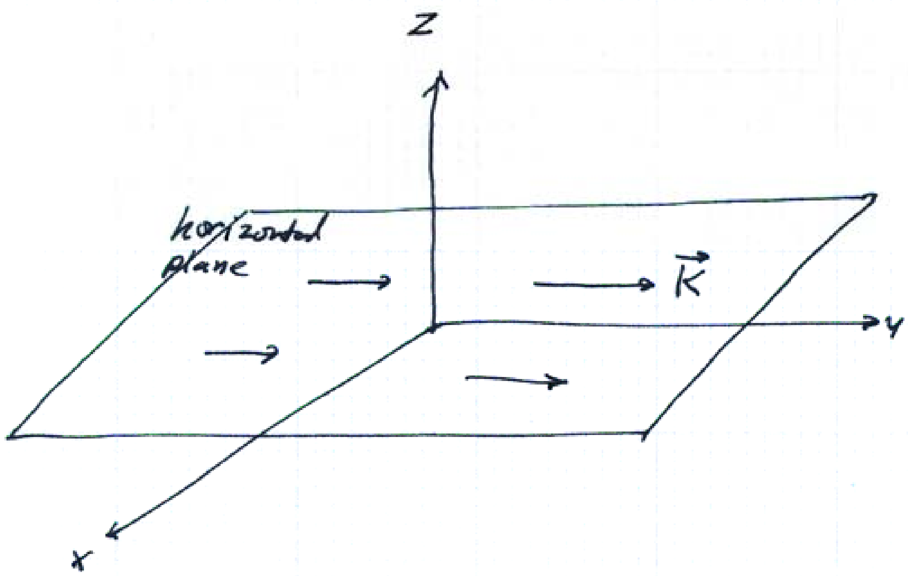
Proof. Because then

$$V(\vec{x},t) = \int G(\vec{x}t\,;\vec{x}'t')\,\frac{\rho(\vec{x}',t')}{\epsilon_0}\, d^3x'\,dt'$$

$$= \frac{1}{\epsilon_0}\int \frac{\rho(\vec{x}', t - |\vec{x}-\vec{x}'|/c)}{4\pi|\vec{x}-\vec{x}'|}\, d^3x'$$ SAME AS ABOVE!

Q.E.D.

Quiz Question



The horizontal plane $z=0$ carries an electric current with surface current density

$$\vec{K}(x,y,t) = K_0\,\hat{j}\,e^{-i\omega t}.$$

(A) Determine the vector potential $\vec{A} = A_y(z,t)\,\hat{j}$.

(B) Determine the magnetic field $\vec{B}(z,t)$.

(N.B. By translation invariance, $\vec{A}$ and $\vec{B}$ do not depend on x or y.)