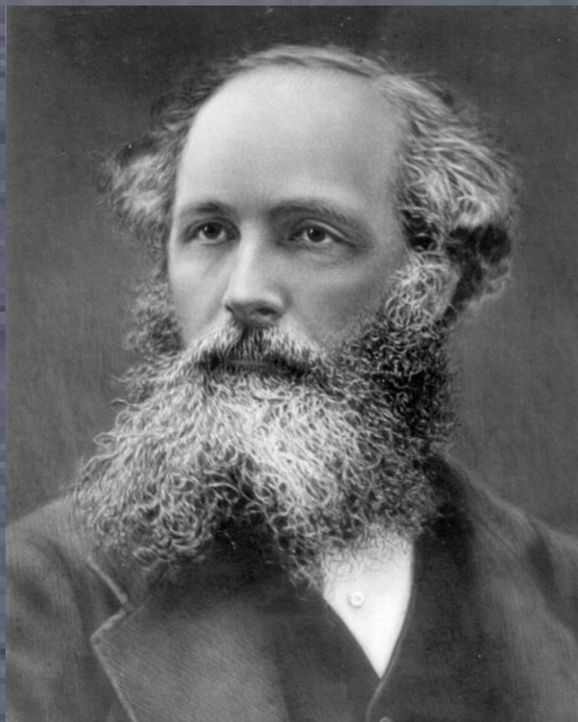


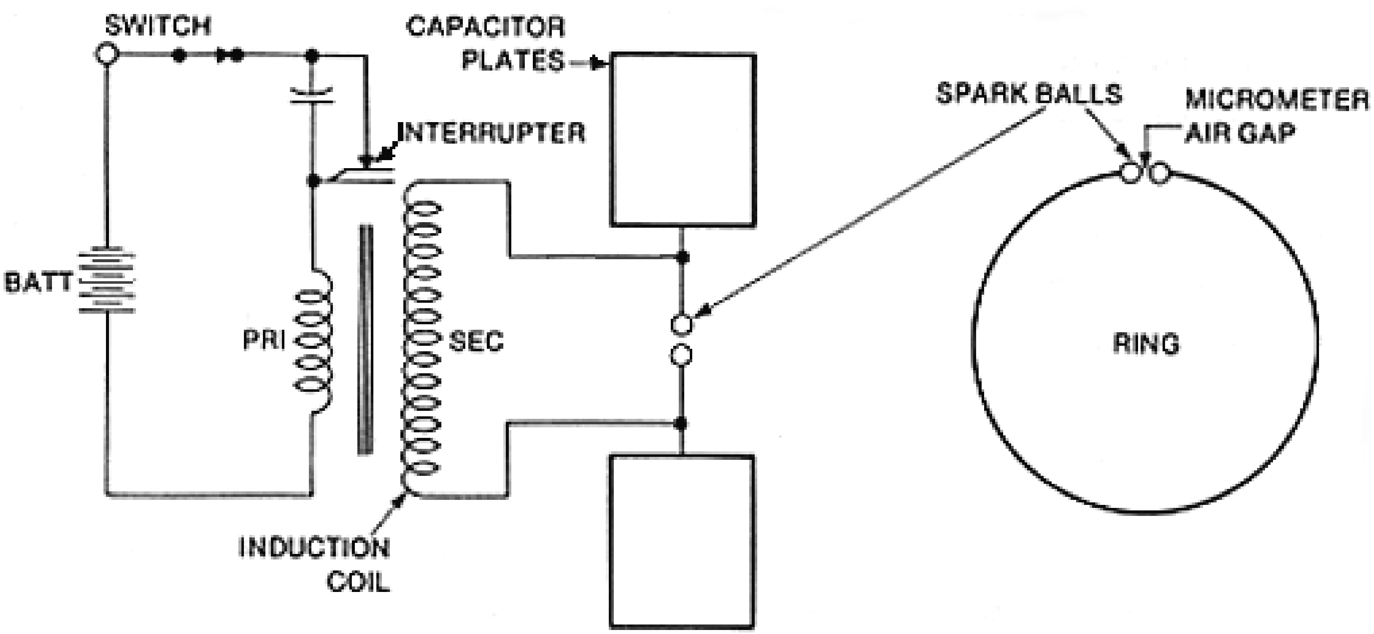
James Clerk Maxwell

Heinrich Hertz

Guglielmo Marconi

Radiation of Electromagnetic Waves

Heinrich Hertz - 1888

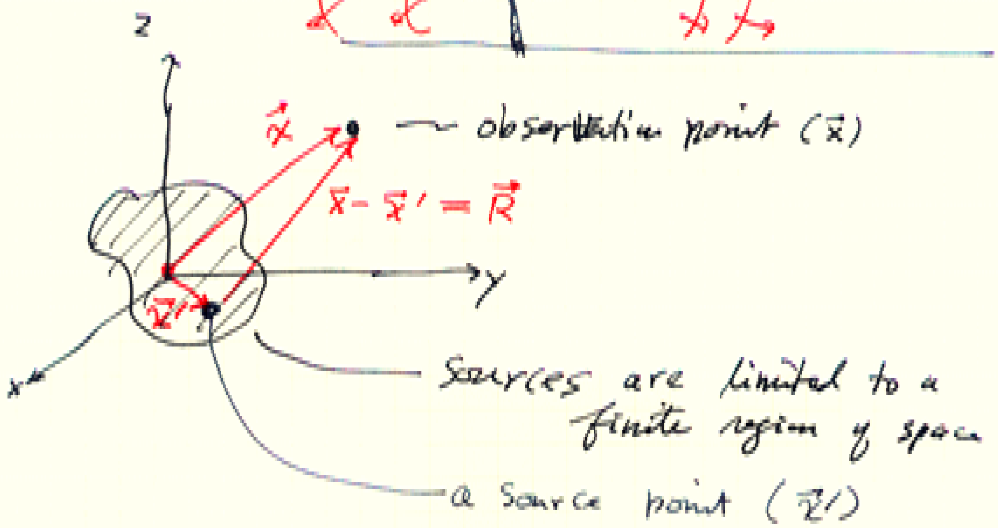


The English mathematical physicist, Sir. Oliver Heaviside, said in 1891, "Three years ago, electromagnetic waves were nowhere. Shortly afterward, they were everywhere."

Electric Dipole Radiation

Given the sources ($\rho(\vec{x},t)$ and $\vec{J}(\vec{x},t)$) what is the radiation?

To make plane waves we would need infinite sources. That's interesting as an academic exercise (homework questions) but not realistic. More important — finite sources make spherical waves.



Using the Lorentz gauge,

$$\vec{A}(\vec{x},t) = \mu_0 \int \frac{\vec{J}(\vec{x}\,', t-R/c)}{4\pi R}\, d^3x'$$

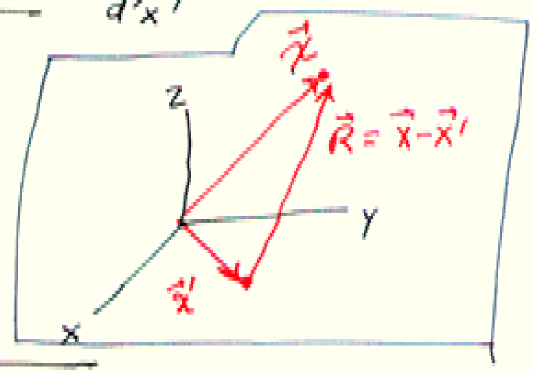
$$V(\vec{x},t) = \frac{1}{\epsilon_0} \int \frac{\rho(\vec{x}\,', t-R/c)}{4\pi R}\, d^3x'$$

and $\nabla\cdot\vec{J} = -\dfrac{\partial\rho}{\partial t}$ ← units: $\frac{1}{m}\,\frac{A}{m^2} = \frac{C/s}{m^3} = \frac{C/m^3}{s}$ ✓

<u>The radiation fields</u> (propagating away from the sources)

↳ i.e., the <u>asymptotic fields</u> as $r \to \infty$

$$\vec{A}(\vec{x},t) = \mu_0 \int \frac{\vec{J}(\vec{x}', t - R/c)}{4\pi R}\, d^3x'$$



Far zone, $r \gg r'$

We may approximate $R \approx r$.

$$R = |\vec{x} - \vec{x}'| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta} \quad \text{(exact)}$$

$$R \approx r - r'\cos\theta + O\left(\frac{r'^2}{r}\right) \quad \text{(improved approximation)}$$

$$R \approx r \quad \text{(lowest approximation)}$$

The lowest approximation is good enough to determine the radiation fields. So, we have

$$\vec{A}(\vec{x},t) \sim \frac{\mu_0}{4\pi r} \int \vec{J}(\vec{x}', t')\, d^3x'$$

where $t' = t - \frac{R}{c} \sim t - \frac{r}{c}$

<u>Theorem</u> $\vec{A}(\vec{x},t) \sim \frac{\mu_0}{4\pi r} \left.\frac{d\vec{p}}{dt}\right|_{t - r/c}$

where $\vec{p}$ is the electric dipole moment of the source.

$$\vec{A} \approx \frac{\mu_0}{4\pi r}\int \vec{J}(\vec{x}', t')\, d^3x' = \frac{\mu_0}{4\pi r}\frac{d\vec{p}}{dt'}$$ R2/3

Proof

Note that $\int \nabla\cdot(x_i \vec{J})\, d^3x = 0$

by Gauss's theorem since $x_i \vec{J} \to 0$ on the surface at ∞. So

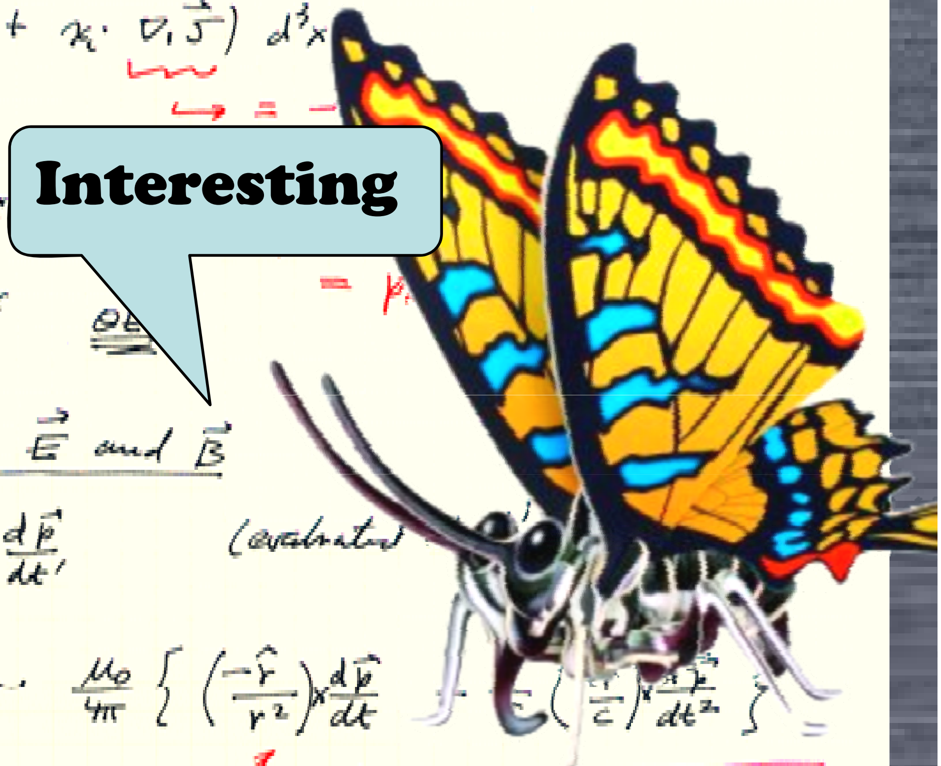
$$0 = \int \left(J_i + x_i \underbrace{\nabla\cdot\vec{J}} \right) d^3x$$

$\to = -$

$$0 = \int J_i\, d^3x$$

$$\int J_i\, d^3x = \frac{dp_i}{dt}$$ QED

$= p$



The asymptotic $\vec{E}$ and $\vec{B}$

$\vec{A} \sim \frac{\mu_0}{4\pi r}\frac{d\vec{p}}{dt'}$ (evaluated

$$\vec{B} = \nabla\times\vec{A} \sim \frac{\mu_0}{4\pi}\left\{ \left(\frac{-\hat{r}}{r^2}\right)\times\frac{d\vec{p}}{dt} \quad \left(\frac{-\hat{r}}{c}\right)\times\frac{d^2\vec{p}}{dt^2} \right\}$$

$\nabla r = \hat{r}$; $\nabla t' = -\frac{\hat{r}}{c}$

This term is negligible as $r \to \infty$.
Neglect $O(1/r^2)$ in $\vec{B}$ or $\vec{E}$.

$$\vec{B} \sim \frac{-\mu_0}{4\pi r c}\,\hat{r}\times\frac{d^2\vec{p}}{dt^2} \equiv \vec{B}_{rad}$$

This is the magnetic part of the radiation field.

Note: $\vec{B}_{rad} \propto 1/r$.

The asymptotic electric field (Lorentz gauge)

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \nabla\cdot\vec{A} = \frac{-1}{c^2}\frac{\partial V}{\partial t}$$

A simpler derivation ...

$$\underbrace{\nabla\times\vec{B}_{rad}} = \frac{1}{c^2}\frac{\partial \vec{E}_{rad}}{\partial t}$$ Displacement Current

$$\hookrightarrow = \nabla\times\left[\frac{-\mu_0}{4\pi r c}\,\hat{r}\times\frac{d^2\vec{p}}{dt^2}\bigg|_{t-r/c}\right]$$

$$\sim \frac{-\mu_0}{4\pi r c}\left(\frac{-\hat{r}}{c}\right)\times\left(\hat{r}\times\frac{d^3\vec{p}}{dt^3}\right)$$

$$\nabla t' = -\frac{\hat{r}}{c}$$

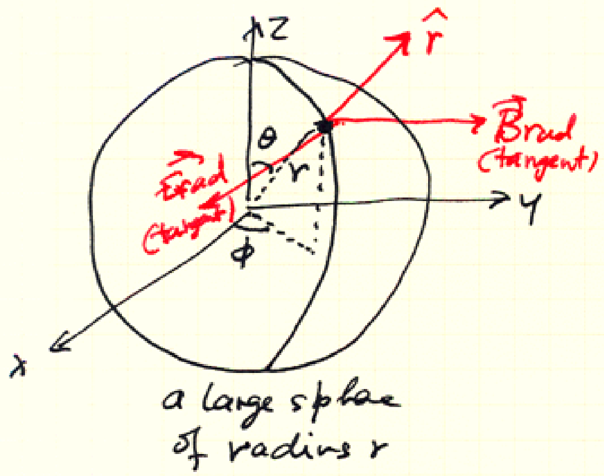
$\Longrightarrow$

$$\vec{E}_{rad} = \frac{\mu_0}{4\pi r}\,\hat{r}\times\left(\hat{r}\times\frac{d^2\vec{p}}{dt^2}\right)\bigg|_{t'=t-r/c}$$

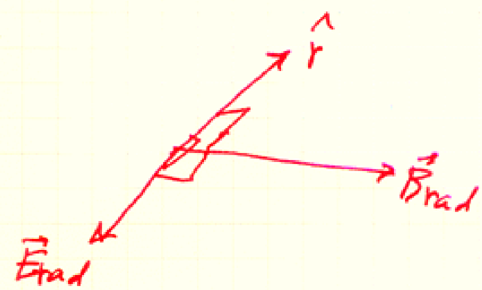
Note $\vec{E}_{rad} \propto 1/r$.

Also, $\vec{E}_{rad} = c\,\vec{B}_{rad}\times\hat{r}$ and $\vec{B}_{rad}\perp\hat{r}$

$\therefore$ $\hat{r}$, $\vec{E}_{rad}$ and $\vec{B}_{rad}$ form an orthogonal triad

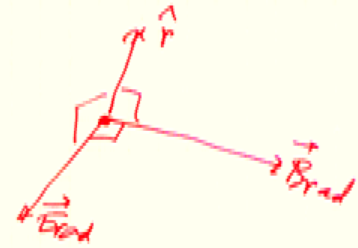


a large sphere of radius r



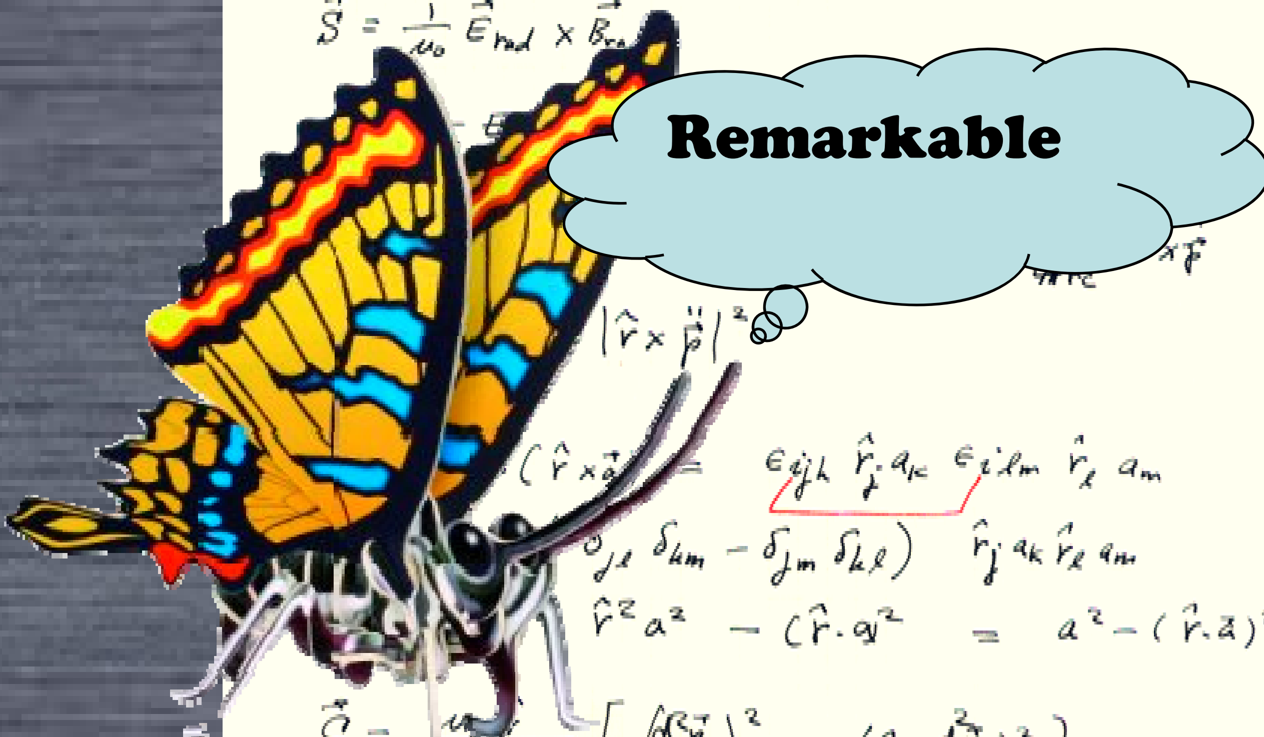
$R_2/5$

Energy flux in radiation



The Poynting vector is

$$\vec S = \frac{1}{\mu_0}\vec E_{rad}\times \vec B_{ra\ldots}$$



$$|\hat r \times \ddot{\vec p}|^2$$

$$(\hat r \times \vec a)^2 = \epsilon_{ijk}\,\hat r_j a_k\,\epsilon_{ilm}\,\hat r_l a_m$$

$$(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\;\hat r_j a_k \hat r_l a_m$$

$$\hat r^2 a^2 - (\hat r\cdot a)^2 = a^2 - (\hat r\cdot \vec a)^2$$

$$\vec S = \frac{\mu_0}{c(4\pi r)^2}\left[\left(\frac{d^2\vec p}{dt^2}\right)^2 - \left(\hat r\cdot\frac{d^2\vec p}{dt^2}\right)^2\right]_{t'=t-r/c}$$

The differential power is

$$\frac{dP}{d\Omega} = \hat r\cdot\vec S\,r^2 = \frac{\mu_0}{16\pi^2 c}\left[\ddot{\vec p}^{\,2} - (\hat r\cdot\ddot{\vec p})^2\right]$$

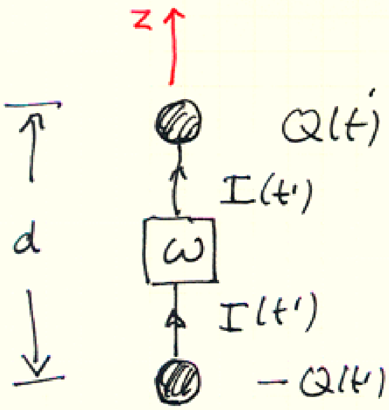
Do you see why $\vec S \propto \frac{1}{r^2}$ is important?
Energy conservation! $\frac{dP}{d\Omega}$ independent of r.

and the total power is

$$P = \int \frac{dP}{d\Omega}\,d\Omega = \frac{\mu_0}{16\pi^2 c}\,|\ddot{\vec p}|^2 \int \underbrace{(1-\cos^2\theta)\sin\theta\,d\theta}_{2-2/3\,=\,4/3}\,\underbrace{d\varphi}_{2\pi}$$

$$P = \frac{\mu_0}{6\pi c}\left|\frac{d^2\vec p}{dt^2}\right|^2 \quad (\text{at } t' = t - r/c)$$

Example: the Hertzian Dipole



$$Q(t') = Q_0 \cos\omega t'$$

$$I(t') = \dot{Q} = -\omega Q_0 \sin\omega t'$$

$$\vec{p}(t') = Q d \hat{k} = \hat{k}\, p_0 \cos\omega t'$$

$$p_0 = Q_0 d$$

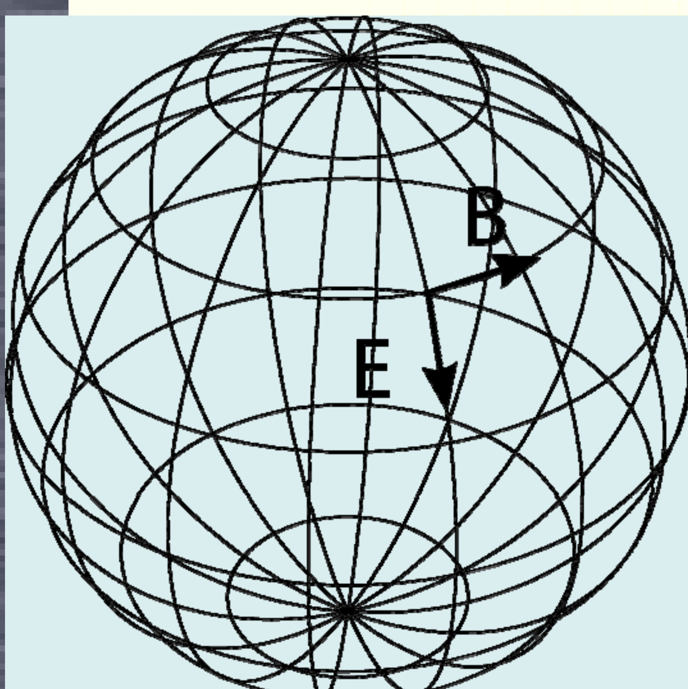
$$\dot{\vec{p}} = -\hat{k}\, \omega p_0 \sin\omega t'$$

$$\ddot{\vec{p}} = -\hat{k}\, \omega^2 p_0 \cos\omega t'$$

$$\vec{B}_{rad} = \frac{-\mu_0}{4\pi r c}\, \hat{r} \times \ddot{\vec{p}} = \frac{-\mu_0 p_0 \omega^2}{4\pi c} \frac{\sin\theta}{r} \hat{\phi} \cos\omega t'$$

$$\vec{E}_{rad} = c \vec{B}_{rad} \times \hat{r} = \frac{-\mu_0 p_0 \omega^2}{4\pi} \frac{\sin\theta}{r} \hat{\theta} \cos\omega t'$$

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{\mu_0 p_0^2 \omega^4}{16\pi^2 c} \frac{\sin^2\theta}{r^2} \hat{r} \cos^2\omega t'$$

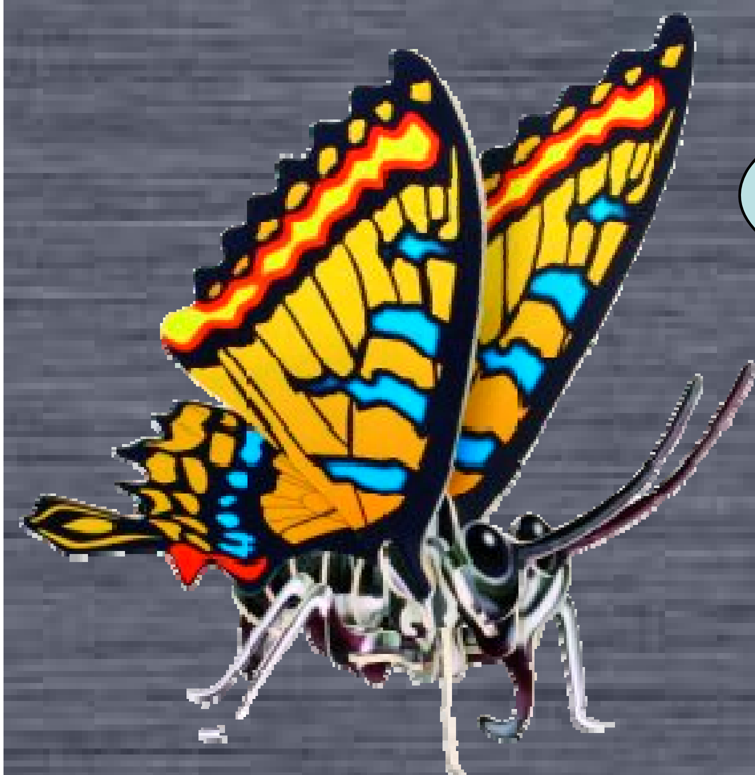


$$P_{\substack{\text{average}\\ \text{over time}}} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$

Quiz Question

Apply the classical theory for Hertzian dipole radiation on an atomic scale. That is, suppose p_0 = e d where d = 10^{-10} m; and suppose $\hbar\omega = \Delta$ where Δ = 10 eV = 10 x $1.6\text{x}10^{-19}$ J. Then:
(A) Calculate the classical radiated power, P_{avg} , in eV/s.
(B) Estimate the classical lifetime of the radiation process.

$$\hbar\omega = \Delta$$



$$\hbar = 1.055 \; x \; 10^{-34} \text{ Js}$$