The Liénard - Wiechert Potentials

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The Liénard-Wiechert Potentials

↳ complete fields of a point charge; ← a particle

a very basic theoretical question, but not always relevant because of quantum physics.

We know, in the Lorentz gauge,

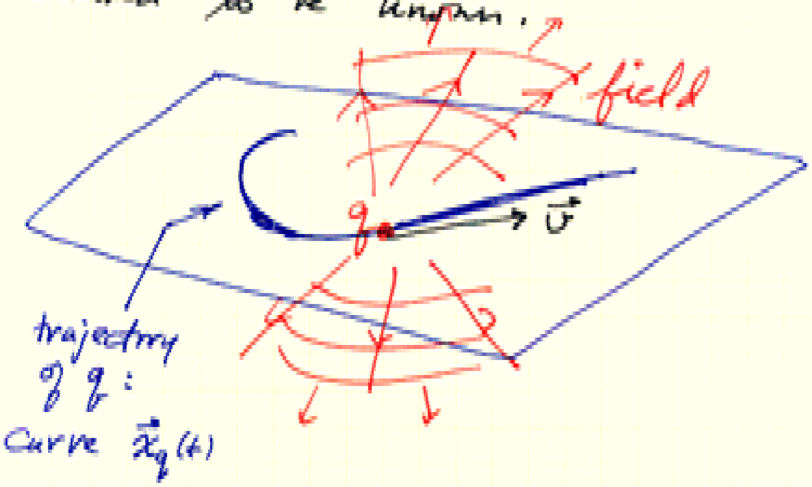
$$V(\vec{x},t) = \frac{1}{\epsilon_0}\int \frac{\rho(\vec{x}', t-R/c)}{4\pi R}\, d^3x'$$

and similarly $\vec{A}(\vec{x},t)$. where $R = |\vec{x}-\vec{x}'|$;

For a point charge, in classical physics,

$$\rho(\vec{x}',t') = q\,\delta^3[\vec{x} - \vec{x}_q(t')].$$

Here $\vec{x}_q(t)$ is the trajectory of q, which is assumed to be known.



Now, it's only math to evaluate

$$V(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\int \delta^3\left[\vec{x}' - \vec{x}_q(t - |\vec{x}-\vec{x}'|/c)\right] \frac{d^3x'}{|\vec{x}-\vec{x}'|}$$

but the calculation requires care!

Example in 1 dimension

$$\mathcal{J} = \int_{-\infty}^{\infty} f(x')\, \delta[x' - g(x')]\, dx'$$

Let x_0 be the point where $x_0 = g(x_0)$. (sharp spike at $x'=x_0$)

Change variable of integration from x' to $\xi = x' - g(x')$.

Note $d\xi = dx'[1 - g'(x')]$ notation $g' = \frac{dg}{dx}$

$$\therefore\ \mathcal{J} = \int_{-\infty}^{\infty} f(x')\, \delta(\xi)\, \frac{d\xi}{|1-g'(x')|} = \frac{f(x_0)}{|1-g'(x_0)|}$$

$\xi = 0$ means $x' = x_0$

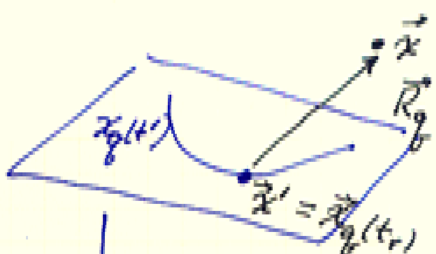
Apply the method to our 3D integral...

$$\mathcal{J}_q = \int \delta^3\left[\vec{x}' - \vec{x}_q\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right] \frac{d^3x'}{|\vec{x}-\vec{x}'|}$$

$$= \frac{1}{|\vec{x} - \vec{x}_q(t_r)|}\ \frac{1}{\mathrm{Det}\left(\frac{\partial \xi_i}{\partial x'_j}\right)} \quad \leftarrow \text{"Jacobian"}$$

- Retarded time t_r

$$x' = \vec{x}_q\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$$



$\vec{x}'$ is a point on the trajectory, with time t_r. The implicit equation for t_r is

$$t_r = t - \frac{|\vec{x} - \vec{x}_q(t_r)|}{c}$$

$\vec{R}_q = \vec{x} - \vec{x}_q(t_r)$

$R_q = |\vec{R}_q|$

$\hat{n} = \vec{R}_q / R_q$

- Jacobian

$$\vec{\xi} = \vec{x}' - \vec{x}_q(t - |\vec{x}-\vec{x}'|/c)$$

$$\mathrm{Jac} = \mathrm{Det}\ \frac{\partial \xi_i}{\partial x'_j}$$

$$\vec{\xi} = \vec{x}' - \vec{x}_q(t - |\vec{x}-\vec{x}'|/c)$$

$$\frac{\partial \xi_i}{\partial x'_j} = \delta_{ij} - \frac{\partial x_{q\,i}}{\partial t} \frac{\partial(t - |\vec{x}-\vec{x}'|/c)}{\partial x'_j}$$

$$= \delta_{ij} - v_i \frac{(x-x')_j}{c|\vec{x}-\vec{x}'|}$$

This is to be evaluated at $\vec{\xi} = 0$; i.e., $\vec{x}' = \vec{x}_q(t_r)$

$$= \delta_{ij} - \beta_i n_j$$

$\vec{\beta} = \frac{\vec{v}}{c}$ and $\hat{n} = \vec{R}_q / R_q$

$$\text{Jacobian} = \mathrm{Det}\left(\frac{\partial \xi_i}{\partial x'_j}\right) = \mathrm{Det}(\delta_{ij} - \beta_i n_j)$$

$$= \begin{vmatrix} 1-\beta_1 n_1 & -\beta_1 n_2 & -\beta_1 n_3 \\ -\beta_2 n_1 & 1-\beta_2 n_2 & -\beta_2 n_3 \\ -\beta_3 n_1 & -\beta_3 n_2 & 1-\beta_3 n_3 \end{vmatrix}$$

Short cut —

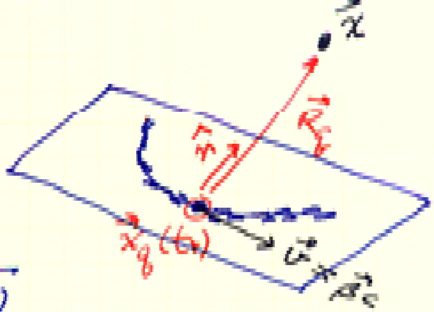
Since it's a scalar, pick a c.s. where $\vec{\beta} = \beta_1 \hat{i}$; i.e. where $\beta_2 = \beta_3 = 0$

Then $Jac = (1-\beta_1 n_1)\cdot 1 + \beta_1 n_2 \cdot 0 - \beta_1 n_3 \cdot 0 = 1 - \beta_1 n_1$

Thus $Jac = 1 - \vec{\beta}\cdot\hat{n}$

<u>Result</u> $$V(\vec{x},t) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{x}-\vec{x}_q(t_r)|} \frac{1}{Jac.}$$

$$V(\vec{x},t) = \frac{q}{4\pi\epsilon_0 R_q} \frac{1}{1-\hat{n}\cdot\vec{\beta}}$$



Similarly,

$$\vec{A}(\vec{x},t) = \frac{\vec{v}}{c^2} V(\vec{x},t) = \frac{\mu_0 q \vec{v}}{4\pi R_q (1-\hat{n}\cdot\vec{\beta})}$$

The fields

$R_6/4$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \nabla \times \vec{A}$$

It's only math, but the calculations of the derivatives are very intricate.

For example, consider ∇V

$$= \nabla \frac{q}{4\pi\epsilon_0 R_q} \frac{1}{1 - \hat{n}\cdot\vec{\beta}}$$

$\vec{R}_q = x - \vec{x}_q(t_r)$

$\vec{\beta} = \frac{v(t_r)}{c}$

$t_r = t - |\vec{x} - x_q(t_r)|/c$ depends on $\vec{x}$

$$= \frac{-q}{4\pi\epsilon_0 R_q^2} \frac{\nabla R_q}{1 - \hat{n}\cdot\vec{\beta}} + \frac{q}{4\pi\epsilon_0 R_q} \frac{(-1)}{(1-\hat{n}\cdot\vec{\beta})^2} \nabla(-\hat{n}\cdot\vec{\beta})$$

We need

$$\nabla R_q = \nabla\{|\vec{x} - x_q(t_r)|\} = \nabla\left\{\sqrt{(\vec{x}-\vec{x}_q)^2}\right\}$$

$$= \frac{1}{2}\left[(\vec{x}-\vec{x}_q)^2\right]^{-1/2} \{2(x-x_q)_i\} \nabla (x-x_q)_i$$

$\sum_{i=1}^{3}$ implied

$\nabla = \hat{e}_j \frac{\partial}{\partial x_j}$

$$= \frac{R_{qi}}{|\vec{R}_q|} \hat{e}_j \left\{ \delta_{ij} - \frac{dx_{qi}}{dt_r} \frac{dt_r}{dx_j} \right\}$$

$$= \hat{n} - \hat{n}\cdot\vec{v} \left[\frac{-\hat{n}}{c(1-\hat{n}\cdot\vec{\beta})} \right]$$

prove it from the implicit equation for t_r

$$= \frac{\hat{n}}{1 - \hat{n}\cdot\vec{\beta}}$$

Et Cetera. After pages of careful calculations the final result is obtained.

Note: $\nabla(\hat{n}\cdot\vec{\beta}) = \frac{d}{dt_r}(\hat{n}\cdot\vec{\beta}) \cdot \nabla t_r = \frac{d(\hat{n}\cdot\vec{\beta})}{dt_r} \frac{(-\hat{n})}{c(1-\hat{n}\cdot\vec{\beta})}$

$\Rightarrow$ a term $\frac{d\vec{\beta}}{dt_r} = \frac{\vec{a}}{c}$ acceleration

The final results are

$$\vec{E}(\vec{x},t) = \frac{q}{4\pi\epsilon_0 R_q^2}\frac{(\hat{n}-\vec{\beta})(1-\beta^2)}{(1-\hat{n}\cdot\vec{\beta})^3} + \frac{q}{4\pi\epsilon_0 c^2 R_q}\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\vec{a}]}{(1-\hat{n}\cdot\vec{\beta})^3}$$

$$\vec{B}(\vec{x},t) = \frac{\hat{n}\times\vec{E}(\vec{x},t)}{c}$$

Comments: 1^{st} term $\propto \frac{1}{R_q^2}$ and depends on $\vec{v}$ "near field"

2^{nd} term $\propto \frac{\vec{a}}{c^2 R_q}$ and also depends on $\vec{v}$ but in a different way "far field"

Example

If $\vec{v}(t) = 0$ [i.e., particle (at rest)] then

$$\vec{E}(\vec{x},t) = \frac{q\hat{n}}{4\pi\epsilon_0 R^2} \qquad \text{where } R = |\vec{x}-\vec{x}_q|$$

$$\vec{B}(\vec{x},t) = 0$$

Radiation: Asymptotic behavior as $r\to\infty$; then $R_q \sim r$

$$\vec{E}_{rad} = \frac{q}{4\pi\epsilon_0 c^2 r}\frac{\hat{r}\times[(\hat{r}-\vec{\beta})\times\vec{a}]}{(1-\hat{r}\cdot\vec{\beta})^3} \quad \left\{ \begin{array}{l}\vec{\beta} \text{ \& } \vec{a} \text{ are} \\ \text{evaluated at } t_r = t - r/c\end{array}\right.$$

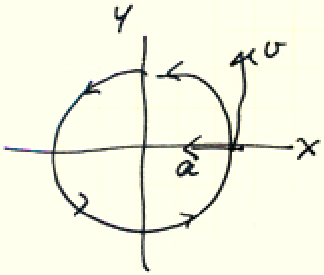
$$\vec{B}_{rad} = \frac{\hat{r}}{c}\times\vec{E}_{rad}$$

$$\vec{S}_{rad} = \frac{\vec{E}_{rad}\times\vec{B}_{rad}}{\mu_0} = \frac{\hat{r}}{\mu_0 c}E_{rad}^2 \qquad \text{because } \vec{E}_{rad}\cdot\hat{r} = 0$$

$$= \frac{\hat{r}\,q^2}{16\pi^2 r^2}\frac{1}{\epsilon_0 c^3}\frac{\{\hat{r}\times[(\hat{r}-\vec{\beta})\times\vec{a}]\}^2}{(1-\hat{r}\cdot\vec{\beta})^6}$$

$$\frac{dP_{rad}}{d\Omega} = r^2\,\hat{r}\cdot\vec{S} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{c^3}\frac{(\hat{r}\times\vec{A})^2}{4\pi(1-\hat{r}\cdot\vec{\beta})^6} \qquad \text{where } \vec{A} = (\hat{r}-\vec{\beta})\times\vec{a}$$

Synchrotron Radiation



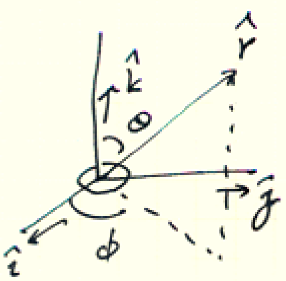
$$\vec{\beta} = \beta\,\hat{j}$$

$$\vec{a} = -a\,\hat{i}$$

$$\vec{A} = (\hat{r} - \vec{\beta}) \times \vec{a} = -a\left[\,\hat{r}\times\hat{i} - \hat{k}\,\right]$$

$$(\hat{r}\times\vec{A})^2 = (\hat{r}\times\vec{A})\cdot(\hat{r}\times\vec{A}) = A^2 - (\hat{r}\cdot\vec{A})^2$$

$$= a^2\left[\,(\hat{r}\times\hat{i})^2 - 2\hat{k}\cdot(\hat{r}\times\hat{i}) + 1\,\right] - a^2(\hat{r}\cdot\hat{k})^2$$



$$\hat{r}\cdot\hat{k} = \cos\theta$$

$$\hat{r} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta$$

$$\hat{r}\times\hat{i} = -\hat{k}\sin\theta\sin\phi + \hat{j}\cos\theta$$

$$(\hat{r}\times\vec{A})^2 = a^2\left\{\sin^2\theta\sin^2\phi + \cos^2\theta + 2\sin\theta\sin\phi + 1 - \cos^2\theta\right\}$$

$$= a^2(\sin\theta\sin\phi + 1)^2$$

$\theta: 0 \to \pi$ $\quad$ $\phi: 0 \to 2\pi$

$$P_{total} = \int \frac{q^2}{4\pi\epsilon_0 c^3}\,\frac{a^2(\sin\theta\sin\phi + 1)^2}{4\pi(1 - \beta\sin\theta\sin\phi)^6}\,\sin\theta\,d\theta\,d\phi$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{2q^2a^2}{3c^3}\,\frac{1}{(1-\beta^2)^2} \qquad \text{where } a = v^2/R_0$$