Gravitational Astrophysics—10 Jan 2012

- Outline
  - Syllabus
  - Gravity in outline
- Syllabus
  - Elementary gravity: Learn gravitational astrophysics without learning the full math. Did Coulomb discover Coulomb's Law from Maxwell's equations?
    Bending of light
    Perihelion shift of Mercury
    Hubble's Law & expansion of the universe
    Geometry of space
    Flux of supernovae
    Inflation
  - Einstein's equations
    Einstein's derivation of his field equation.
    Tensors
    Curvature tensor
    Stress-energy tensor
  - Other topics
    Gravitational waves
    Black holes

For what cases does Einstein's gravity differ from Newton's gravity

- **Questions (One group initiates the answer. We discuss. Another group is responsible for emailing the answer to the entire class.)**

  1. Why is Ast860 called gravitational *astrophysics*? In Ph183, why was the topic of gravity not called gravitational *astrophysics*?
  2. What parameter determines whether Einstein's gravity applies?

Metric

Einstein’s field equation is

$$\boldsymbol{G} = 8\pi G \boldsymbol{T}$$

where $\boldsymbol{G}$ is the Einstein curvature tensor and $\boldsymbol{T}$ is the stress-energy tensor.

For a system with mass and pressure,

$$\boldsymbol{T} = \begin{pmatrix} \text{mass density} & 0 & 0 & 0 \\ 0 & P_x/c^2 & 0 & 0 \\ 0 & 0 & P_y/c^2 & 0 \\ 0 & 0 & 0 & P_z/c^2 \end{pmatrix}$$

Roughly, Einstein's field equation says "mass and pressure cause curvature." Planets feel the curvature and accelerate.

The effects of gravity show up in the curvature of space and distortion of time. We need to calculate these. This is the purpose of the metric.

The metric describes the distance between two nearby points.

3-dimensional

The distance between $\boldsymbol{x}$ and $\boldsymbol{x} + d\boldsymbol{x}$ in rectangular coordinates is

$$ds^2 = dx^2 + dy^2 + dz^2$$

In 3-d cylindrical coordinates (r, θ, ϕ),

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Q: Compute the circumference and radius of a circle at $\theta = \pi/2$ centered at r=0.

4-dimensional (t, x, y, z)

The distance between two events (t, x, y, z) and $(t + dt, x + dx, y + dy, z + dz)$ is

$$ds^2 = -c^2\, dt^2 + dx^2 + dy^2 + dz^2$$

From now on, we will drop the speed of light c. Time and space are measured with the same units.

$$\boxed{ds^2 = -dt^2 + dx^2 + dy^2 + dz^2}$$

Q: If space is measured in cm, what is the time unit in CGS?

Q: In this class, there is no need to copy the equations because you have the slides on paper and in angel (links are on the syllabus). However, you need to understand the equations. What do you need to note about the boxed equation?

Example: A light pulse goes from event A at $(t, x) = (0, 0)$ to nearby event B at $(t, x) = dt\,(1, 1)$. What is the distance between events A and B?

The distance between the two events is

$$ds = \left(-dt^2 + dx^2\right)^{1/2}$$

Since light travels at the speed of light, $dt = dx$. Therefore the distance $ds = 0$.

Schwartzschild metric

The Schwartzschild metric applies outside a star of mass M. The metric is

$$ds^2 = -\left(1 - 2\,\frac{G\,M}{r\,c^2}\right) c^2\, dt^2 + \left[\left(1 - 2\,\frac{G\,M}{r\,c^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Written in geometrical units,

$$\boxed{ds^2 = -\left(1 - 2\,\frac{M}{r}\right) dt^2 + \left[\left(1 - 2\,\frac{M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]}$$

The geometrical mass of the sun is

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In[611]:= Convert[GravitationalConstant SolarMass / SpeedOfLight^2, Kilo Meter]
Out[611]= 1.47713 Kilo Meter
```

Recall $\mathrm{GM}/(r\,c^2)$ is the strength of gravity. At the surface of the sun, the strength of gravity is

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In[609]:= Convert[GravitationalConstant SolarMass / SpeedOfLight^2 / SolarRadius, 1]
Out[609]= 2.12235 × 10^-6
```

Is space curved?

Compute the circumference of a circle centered on the sun and passing through $(r_0, \pi/2, 0)$

Since $dt = 0$, $dr = 0$, and $d\theta = 0$,

$$C = 2\,\pi\, r_0$$

Compute the radial distance from $(r_0, \pi/2, 0)$ to $(2\,r_0, \pi/2, 0)$.

Since $dt = 0$, $d\phi = 0$, and $d\theta = 0$,

$$R = \int_{r_0}^{2\,r_0} (1 - 2\,M/r)^{-1/2}\, dr$$
$$\approx \int_{r_0}^{2\,r_0} (1 + M/r)\, dr$$
$$= r + M \log r\Big|_{r_0}^{2\,r_0} = r_0 + M \log 2$$

Radial distance is long by $M \log 2 =$

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In[612]:= 1.48 Log[2] Kilo Meter
Out[612]= 1.02586 Kilo Meter
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Lessons from special relativity

Special relavivity does not have gravity, but its techniques are useful for gravity.

- Use events and distances to calculate physics. Think about events first. Then apply coordinates.
- Speed of light is the same for all inertial frames.
- The distance between two events is

$$ds^2 = -dt^2 + dx^2$$

- The distance does not depend on the frame of reference.

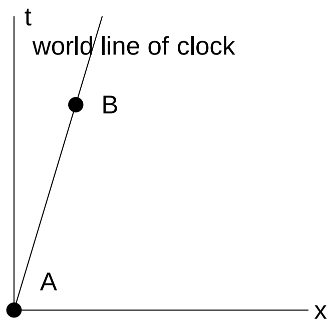
Example: time dilation

A clock tics every τ. The clock is moving at speed v. How often does the clock tic for me, who is stationary?

First choose events.

Event A: clock tics. Event B: clock tics next.

Out[670]=



Think about distances

For the clock, the distance^2 between the two events is

$$ds^2 = -\tau^2$$

For me, the distance^2 between event A at $(0, 0)$ and event B at (t, x) is

$$ds^2 = -t^2 + x^2$$

Since distance is invariant,

$$\tau^2 = t^2 - x^2.$$

Since $x = v\,t$,

$$t = \tau\left(1 - v^2\right)^{-1/2}.$$

In my frame, more time elapsed. For $v = 0.99$, t/τ is

In[613]:=
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1 / Sqrt[1 - .99^2]
```

Out[613]= 7.08881