Initialization

Orbits & paths of light rays—26 Jan

- Announcements
  - Homework 2 is due on 1/31. The link is on the syllabus on angel.
  - We start cosmology on Tues. We are ahead of schedule. Read about the Robertson-Walker metric
- Outline
  - Orbits that are like Newtonian ones
  - Orbits that are not like Newtonian ones
  - Orbit for a light ray

Differences between Newtonian orbits and orbits with General Relativity

Recall:

(1)The length^2 of the 4-velocity is -1.

(2) u_0 is conserved because the metric is independent of time.

(3) u_ϕ (in the ϕ direction) is conserved because the metric is independent of ϕ.

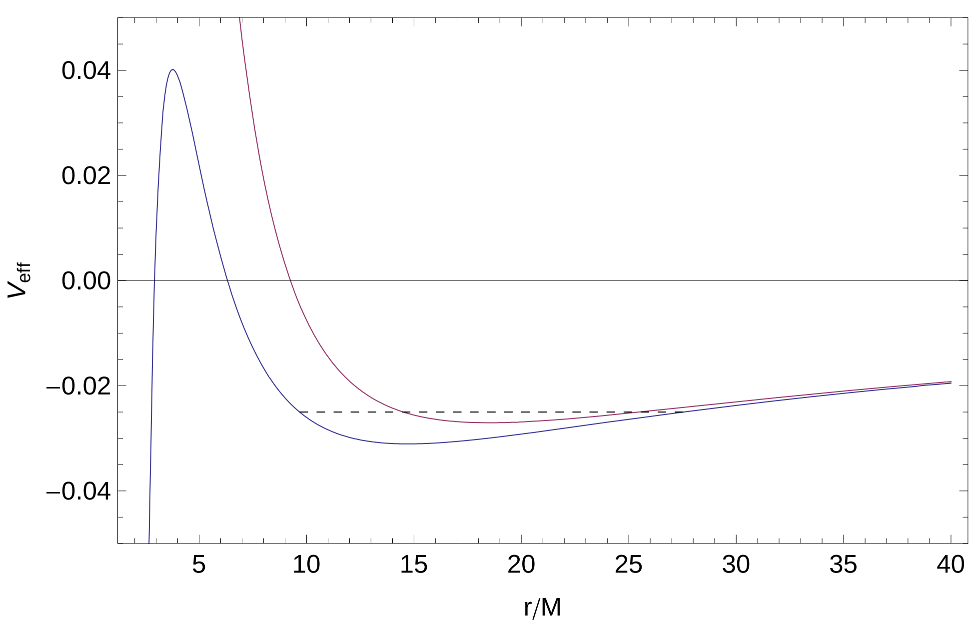
From (1–3), we derived

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{d\,r}{d\,\tau}\right)^2 + V_{\text{eff}}(r),$$

where $V_{\text{eff}}(r) = -\frac{M}{r} + \frac{l^2}{2\,r^2} - \frac{M\,l^2}{r^3}$

Scale r and angular momentum l by dividing by M.

▪ When orbits are possible



Caption: Effective potential for Einstein's theory (blue) and for Newton's theory (purple). The dotted line is $\frac{1}{2}\left(e^2-1\right)$.

Orbit exists if $\frac{e^2-1}{2} > V_{\text{eff}}(r)$ for some r, and there are two turning points. The turning points are where $\frac{e^2-1}{2} = V_{\text{eff}}(r)$.

- **Values of energy and angular momentum for Earth**

Radius of the orbit in units of M_{sun}.

r/M

```
Convert[AstronomicalUnit / mSun, 1]
```

1.01276×10^8

Angular momentum in units of M_{sun}

$l^2 = M\, a\left(1-\epsilon^2\right)$

$l/M = \left[a\left(1-\epsilon^2\right)/M\right]^{1/2}$

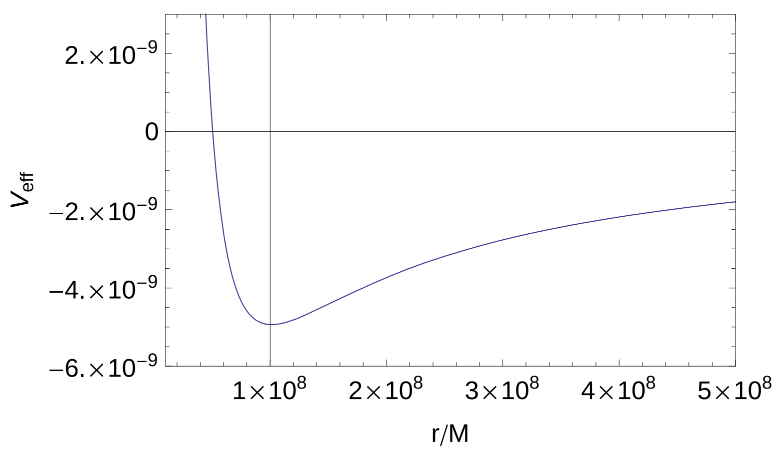
```
l = Sqrt[Convert[AstronomicalUnit / mSun, 1]]
```

10 063.6

Energy $\mathcal{E} = \frac{1}{2}\left(e^2-1\right) = -\frac{1}{2}\,\frac{M}{a}$

```
e1 = -Convert[mSun / 2 / AstronomicalUnit, 1]
```

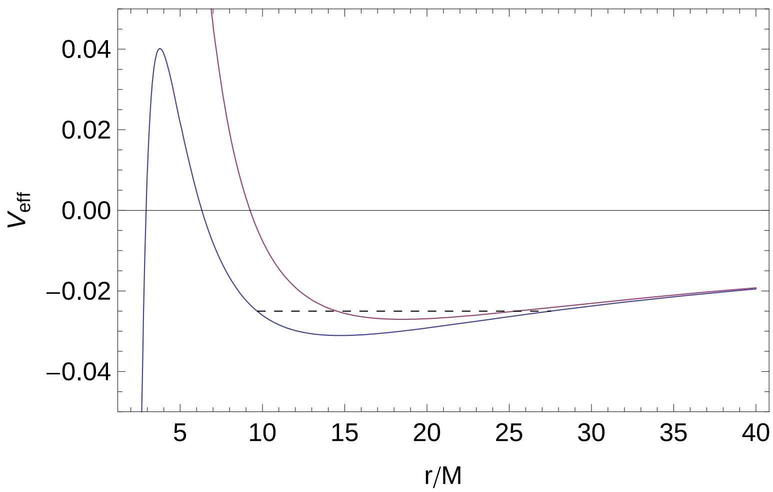
-4.93701×10^{-9}



Caption: $V_{\text{eff}}(r)$ for Earth’s orbit. The energy $0.8\,\mathcal{E}$ is shown to make it visible. The dimensionless angular momentun is 10063.6.

Q: For what energy or range of energies is the orbit elliptical? Parabolic? Hyperbolic?

- **New orbits in General Relativity**



Caption: Caption: Effective potential for Einstein’s theory (blue) and for Newton’s theory (purple). The dimensionless angular momentun is 4.3.

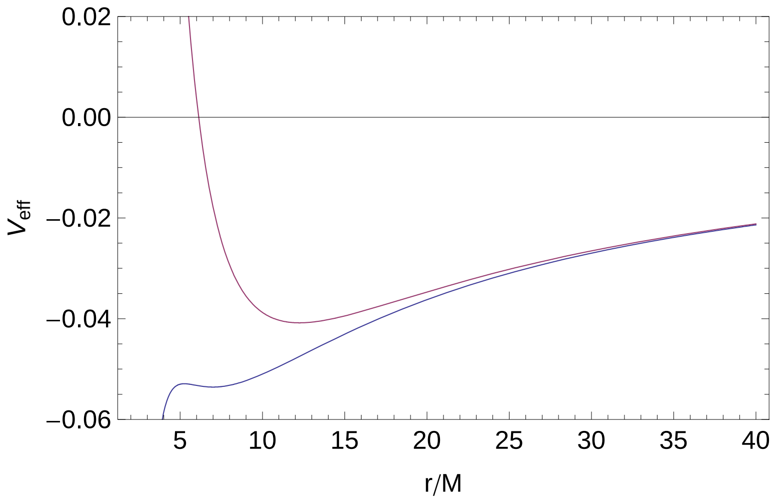
We already know that the shapes of the orbits are not the same as in the Newtonian case.

Q: What is a new kind of orbit that is not possible with the Newtonian theory?

- **Plots**

■ When orbits are not possible

```
plotVEffective[3.5, -.053, PlotRange → {-.06, .02}]
```



Caption: Caption: Effective potential for Einstein’s theory (blue) and for Newton’s theory (purple). The dimensionless angular momentum is $l/M = 3.5$.

Find extrema of $V_{\text{eff}}(r)$

```
Solve[D[vEff[r, l], r] == 0, r]
```

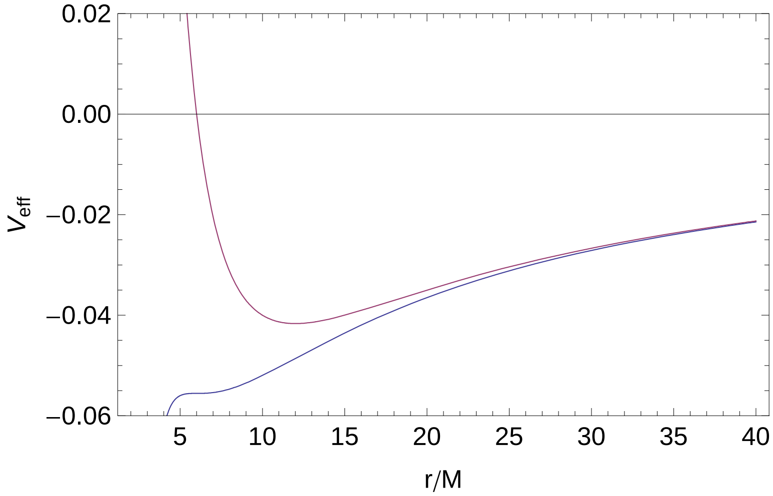
$$\left\{\left\{r \to \frac{1}{2}\left(l^2 - l\sqrt{-12 + l^2}\right)\right\}, \left\{r \to \frac{1}{2}\left(l^2 + l\sqrt{-12 + l^2}\right)\right\}\right\}$$

If $l/M > 12^{1/2}$, there are two extrema. If $l/M < 12^{1/2}$, the extrema are imaginary.

```
Sqrt[12.]
```

3.4641

```
plotVEffective[Sqrt[12], -.053, PlotRange → {-.06, .02}]
```



Q: You are near a black hole. What do you do to fall into the black hole? What do you do to prevent falling into the black hole?

Particles without mass. Light

For planets, we used

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\phi}{d\tau}\right).$$

We could just as well have used 4-momentum $p^\mu = m\,u^\mu$.
For photons, that is not valid because $d\tau$ is 0.

Instead of proper time, use a parameter λ. Then

$$p^\mu = \left(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, \frac{d\theta}{d\lambda}, \frac{d\phi}{d\lambda}\right).$$

We will solve a particular orbit and find out what λ is in that case.

Recall:
(1) The length of the 4-momentum p is 0.
(2) p_0 is conserved because the metric is independent of time. Define

$$e = p_0 = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda}.$$

(3) p_ϕ (in the ϕ direction) is conserved because the metric is independent of ϕ.
Define

$$l = r^2 \frac{d\phi}{d\lambda}$$

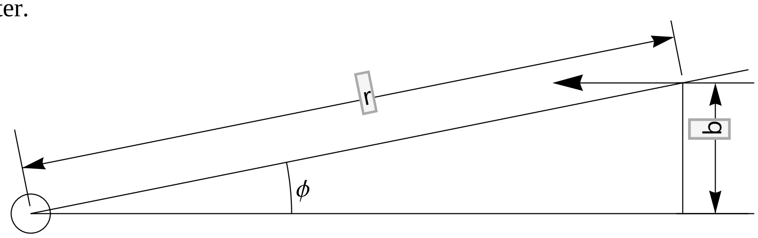
From (1–3), we get

$$-\left(1-\frac{2M}{r}\right)^{-1} e^2 + \left(1-\frac{2M}{r}\right)^{-1} \left(\frac{dr}{d\lambda}\right)^2 + \frac{l^2}{r^2} = 0.$$

$$\left(\frac{e}{l}\right)^2 = \frac{1}{l^2} \left(\frac{dr}{d\lambda}\right)^2 + W_{\text{eff}}(r),$$

where $W_{\text{eff}}(r) = \frac{1}{r^2}\left(1-\frac{2M}{r}\right)$.

A photon is headed for $r = b$ from the star. The parameter b is called the impact parameter.



$\sin\phi = b/r$, and $\cos\phi \frac{d\phi}{d\lambda} = -\frac{b}{r^2} \frac{dr}{d\lambda}$. Rewrite as

$$\cos\phi\, l = -b \frac{dr}{d\lambda}$$

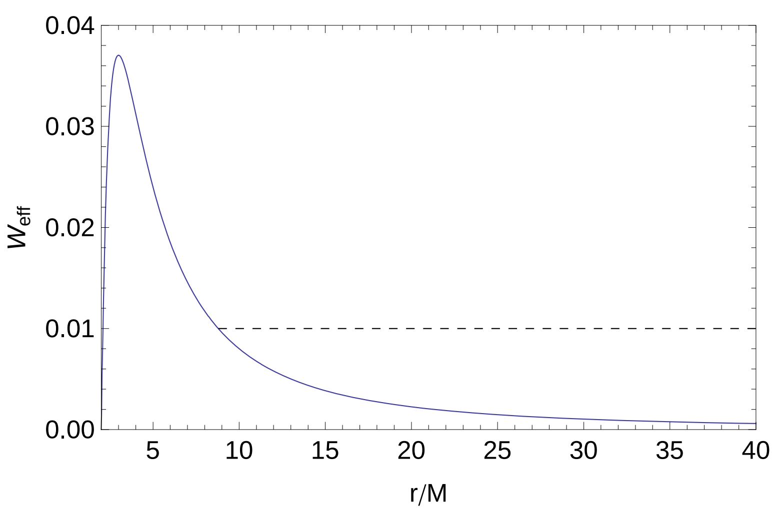
As $r \to \infty$, LHS $\to l$.

At $r \to \infty$, $e = \frac{dt}{d\lambda}$. Since the length of the 4-momentun of a photon is 0, $\frac{dt}{d\lambda} = \frac{dr}{d\lambda}$ at $r \to \text{inf}$. Therefore

$$e \to \frac{dr}{d\lambda}$$

RHS $\to b\, e$. Therefore $b = l/e$.

$$\boxed{\frac{1}{b^2} = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{\text{eff}}(r)}$$



Caption: $W_{\text{eff}}(r)$ for $b = 10\,M$.

Q: When $b = 10\,M$, photons go to in to $r = 8.6\,M$ and then back out again. What is the path of the photon in space?

Find the peak of W_{eff}.

```
r /. Solve[D[wEff[r], r] == 0, r][[1]]
```

3

Solve for b

```
b = Sqrt[1 / wEff[%]]
```

$3\sqrt{3}$

If $b < 27^{1/2} M$, photons go toward the mass and never go back out.

If the sun were a point mass, then the critical impact parameter for capture is

```
Sqrt[27] mSun
```

7675.41 Meter

- **Plots**

What is the parameter λ? Path of a radial light ray

$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda}$ and $l = r^2 \frac{d\phi}{d\lambda}$. What is λ?

The orbit equation is

$$\left(\frac{e}{l}\right)^2 = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{\text{eff}}(r),$$

where

$$W_{\text{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right).$$

We have differential equations for $\frac{dr}{d\lambda}$ and $\frac{d\phi}{d\lambda}$, which we can solve. Do the simple case of an almost radial light ray, for which $l/r \ll 1$. In that case $\left(\frac{e}{l}\right)^2 = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{\text{eff}}(r)$ becomes

$$e^2 = \left(\frac{dr}{d\lambda}\right)^2.$$

The solution is

$$r = e\lambda.$$

Surprise: the parameter λ is not time. For a radial light ray, the parameter λ is the radial coordinate divided by the energy at ∞. (Recall e is the energy at $r \to \infty$.)

Calculate the coordinate time.

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda}$$
$$= \left(1 - \frac{2M}{r}\right)\frac{dt}{dr}\frac{dr}{d\lambda}$$

For radial paths, $e^2 = \left(\frac{dr}{d\lambda}\right)^2$. Substitute to get

$$dt = \pm dr\left(1 - \frac{2M}{r}\right)^{-1}$$
$$= \pm dr\left(1 + \frac{2M}{r-2M}\right)$$

Use + for outgoing paths and – for incoming paths. Substitute $r = e\lambda$ to get

$$\Delta t = e\left(\lambda_2 - \lambda_1 + 2M \log\frac{\lambda_2 - 2M/e}{\lambda_1 - 2M/e}\right)$$

If the energy of the photon is bigger, the parameter λ changes more slowly as r and t change. However the path $r(t)$ is independent of energy.