Cosmological measurements—7 Feb

- Announcements
  - Midterm is Tues, the 14th. Midterm from 2010 is on angel. (Link is on the syllabus.) Covers material through last week.
  - Late homework 3 will be accepted on Thurs, but no later. Answers will be posted on Thurs.
- Outline
  - What is the mass density of the universe?

Q: Why is this an important question?

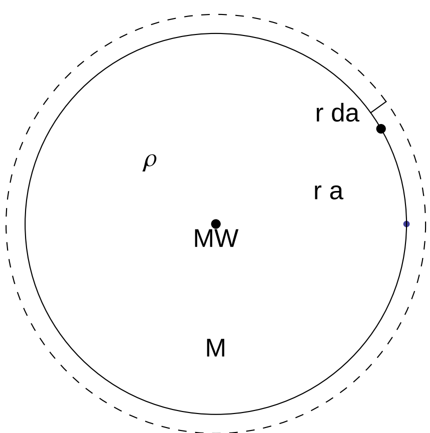
Q: Why is directly measuring the mass in a cubic meter difficult, if not impossible?

  - First Law of thermodynamics for matter with and without pressure.
  - The “force” of matter on a galaxy and on the expansion of the universe. The force of radiation and the force of the cosmological constant.
  - The comoving coordinate vs expansion parameter $r(a)$.

The comoving coordinate determines the outcome of geometrical measurements. E.g., the angle subtended by an object of width L is $\theta = L/r(a)$.
Q: Why do we want to know it as a function of the expansion parameter?

  - The angle subtended by a ruler

"Matter" with and without pressure



The energy inside the sphere changes as the universe expands because the inside does work on the outside. Energy density is ρ.

$$d E = -p A d x = -p d V$$
$$d\left(\rho a^3\right) = -p d a^3.$$

■ Pressureless matter

The speed of galaxies is 300km/s=0.001. The pressure is 10^{-6}.
Q: The pressure of galaxies is small compared with what?

$$d\left(\rho a^3\right) = 0$$

$\rho \sim a^{-3}$

- **Radiation has pressure**

For a photon, $E^2 = p^2 + m^2$ becomes $E = |p|$. The pressure is proportional to the momentum p_x times the rate at which photons are hitting $v_x = p_x / E$. Here, p is momentum. The pressure in one direction is $p_x^2 / E = \frac{1}{3} E$.

Therefore (p is pressure now)

$$p = \frac{1}{3} \rho$$

Put that in

$$d(\rho a^3) = -p\, d a^3 = -\frac{1}{3} \rho\, d a^3$$
$$a^3\, d\rho + \rho\, d a^3 = -\frac{1}{3} \rho\, d a^3$$
$$\frac{d\rho}{\rho} = -\frac{4}{3}\, d a^3 / a^3 = -4 \frac{da}{a}$$
$$d \log \rho = -4\, d \log a$$

to get

$\rho \sim a^{-4}$

Q: Simplicio says, “Radiation can be considered to be photons. I can think of photons as marbles. The energy density should then scale as a^{-3}, just as it does for marbles.” In what way is Simplicio wrong?

- **Vacuum**

The vacuum has energy density from the quantum mechanics. The simpliest model of a field is a harmonic oscillator. The ground state of which is $\frac{1}{2} \hbar \omega$. Each field has some "vacuum energy." This energy density is a property of the field and therefore does not depend on the expansion of the universe.

$$\rho \sim \text{constant}$$

Then the pressure is negative.

$$d\rho\, a^3 = \rho\, d a^3.$$
$$\rho\, d a^3 = -p\, d a^3$$
$$p = -\rho$$

The pressure of the material inside the sphere is pulling on the outside, not pushing.

For pressureless matter

$$\rho \sim a^{-3}.$$

For radiation

$$\rho \sim a^{-4}.$$

For the vacuum

$$\rho \sim \text{constant}.$$

- **Fig**

How the universe expands

Friedman's equation relates the expansion parameter $a(t)$, matter density $\rho(t)$, and the radius of curvature r_0. The position of a galaxy at comoving coordinate r is $r\,a(t)$.

$$\left(\frac{1}{a}\,\frac{d\,a}{d\,H_0\,t}\right)^2-\frac{8\,\pi\,G}{3\,H_0^2}\,\rho=-\frac{1}{H_0^2\,r_0^2\,a^2}$$

Outline

• Interpret Friedman's equation and dE = −pdV as the force of gravity. What replaces Newton's $F=-G\,M\,m\big/X^2$?

• Measuring the universe

The expansion parameter is observed directly. (The univserse expands by the same factor as the wavelength of light.)

Measuring involves distances and angles, which involve $(t,\ r,\ \theta,\ \phi)$.

We need to derive $r(a)$, the comoving coordinate of a galaxy seen at expansion parameter a.

What replaces Newton's law of gravity when applied to the expansion of the universe?

What replaces Newton's law of gravity when applied to the expansion of the universe? We already know what it is for the motion of planets areound a star: Add a term $-M/r\times l^2\big/r^2$ to the energy.

Recall Friedman's equation

$$\left(\frac{da}{dt}\right)^2-\frac{8\,\pi\,G}{3}\,\rho\,a^3\,a^{-1}=-\frac{1}{r_0^2},$$

which we derived partly by considering the energy of a galaxy at some distance from us.

Recall the First Law of thermodynamics

$$d\,E=-p\,d\,V,$$

which says the energy in some region changed by the work done on it.

Rewrite

$$d\,\rho\,a^3=-p\,d\,a^3$$

Differentiate F's equation

$$2\,\frac{d\,a}{d\,t}\,\frac{d\,a}{d\,t^2}=\frac{8\,\pi\,G}{3}\left(a^{-1}\,\frac{d}{d\,t}\,\rho\,a^3-a^{-2}\,\frac{d\,a}{d\,t}\,\rho\,a^3\right)$$

Insert 1st law to get

$$\frac{d^2a}{dt^2}\,\frac{da}{dt}=\frac{4\,\pi\,G}{3}\left(-p\,a^{-1}\,\frac{d}{d\,t}\,a^3-a^{-2}\,\frac{d\,a}{d\,t}\,\rho\,a^3\right)$$

$$\frac{d^2\,a}{d\,t^2}=-G\,\frac{4\,\pi\,a^3}{3}\,(\rho+3\,p)\,\frac{1}{a^2}$$

Since a galaxy is at fixed r, its distance from us is $x=r\,a$.

$$\boxed{\frac{d^2x}{dt^2}=-G\,\frac{4\,\pi\,x^3}{3}\,(\rho+3\,p)\,\frac{1}{x^2}}$$

Q: Interpret the term $\frac{4\,\pi\,x^3}{3}\,\rho$.

Q: In words, the acceleration of gravity is $-G\,M\big/x^2$. What is M?

What replaces Newton's law of gravity for pressureless matter?

$$\frac{d^2 x}{d t^2} = -G \frac{4 \pi x^3}{3} (\rho + 3 p) \frac{1}{x^2}$$

Pressureless matter means the speeds are much less than the speed of light.

$\frac{4 \pi x^3}{3} \rho$ is the mass in a sphere. (Newton found that the force exerted by a spherically distributed extended mass is the same as if all of the mass were placed at the center of the sphere.) The force of gravity on mass m is

$$F = -G (V \rho) m / x^2$$

What if the matter has some pressure? We can guess. The "mass" density is should be replaced by the mass-energy density. A particle of mass μ moving with speed v has energy

$$\mu \left(1 - v^2\right)^{-1/2}$$

The find the mass-energy density, add up $\mu \left(1 - v^2\right)^{-1/2}$ for all of the particles.

$$\rho = n \mu \left(1 - v^2\right)^{-1/2}.$$

Now we have to find the pressure. The pressure in the x-direction is the transport of x-momentum in the x-direction. The momentum transfer in $d t$ over area A is

$$n \frac{\mu v_x}{\left(1-v^2\right)^{1/2}} (A v_x d t)$$

Therefore the pressure is

$$p = n \frac{\mu v_x^2}{\left(1-v^2\right)^{1/2}}$$

If the system is isotropic,

$$p = \frac{1}{3} n \frac{\mu v^2}{\left(1-v^2\right)^{1/2}}$$

Then the force is

$$F = -G V n \mu \left(1 + v^2\right) \left(1 - v^2\right)^{-1/2} m / x^2$$

Q: If the galaxies in the universe are hot, the gravity is bigger. Why?

What replaces Newton's law of gravity for radiation?

$$\frac{d^2 x}{d^2 t} = -G \frac{4 \pi x^3}{3} (\rho + 3 p) \frac{1}{x^2}$$

For thermal radiation, the energy density is $4 \sigma / c T^4$, and the mass density is

$$\rho = 4 \sigma / c^3 T^4$$
$$= 2.1 \times 10^{-33} \text{ kg m}^{-3} \text{ K}^{-4} T^4$$

The pressure is

$$p = \frac{1}{3} \rho.$$

The energy density is $u = \rho c^2$, and in conventional units, $p_{\text{cu}} = p c^2$.

$$\frac{d^2 x}{d \ t^2} = -G \frac{4 \pi x^3}{3} (\rho + 3 p) \frac{1}{x^2}$$
$$F = -G V (2 \rho) m \frac{1}{x^2}$$

What replaces Newton's law of gravity for the vacuum?

$$\frac{d^2 x}{d t^2} = -G \frac{4 \pi x^3}{3} (\rho + 3 p) \frac{1}{x^2}$$

The vacuum has energy density from the quantum mechanical principle $\Delta E \Delta t > h$. Each field has some "vacuum energy." This energy density is a property of the field and therefore does not depend on the expansion of the universe.

Recall $p = -\rho$.

$$F = -G V (\rho + 3 p) m \frac{1}{x^2}$$
$$= +G V (2 \rho) m \frac{1}{x^2}$$

Q: What is unusual about this force?

Radial coordinate of a galaxy

A galaxy emits some light when the expansion parameter was a, and we see the light now when $a = 1$. What is its radial coordinate?

The metric is

$$ds^2 = -dt^2 + a(t)^2 \left[dr^2 / \left(1 - (r/r_0)^2\right) + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right].$$

For the light pulse, $ds = 0$.

$$\int_{\text{emit}}^{\text{recep}} a(t)^{-1} \, d t = r_0 \int_0^{r/r_0} \left(1 - x^2\right)^{-1/2} d x.$$

RHS is

$$r_0 \arcsin (r/r_0)$$

where $(r_0 H_0)^{-2} = \Omega_0 - 1$.

Work on LHS:

Recall the density parameter $\Omega_0 = \frac{8 \pi G}{3 H_0^2}$. Scale time by the Hubble time $\tau = H_0 t$

$$\left(\frac{da}{dH_0 t}\right)^2 - \frac{8 \pi G}{3 H_0^2} \rho a^2 = -\frac{1}{H_0^2 r_0^2}$$

$$\left(\frac{da}{H_0 dt}\right)^2 - \Omega_0 \frac{\rho}{\rho_0} a^2 = 1 - \Omega_0$$

For pressureless matter, $\rho = \rho_0 a^{-3}$.

$$\left(\frac{da}{H_0 d\tau}\right)^2 - \Omega_0 a^{-1} = 1 - \Omega_0$$

$$a(t)^{-1} d t = a(t)^{-1} \frac{dt}{da} da$$

In general this must be integrated numerically. Let's find it for pressureless matter.

Put in Friedman's eqn. to get

$$\text{LHS} = H_0^{-1}\left[a^2(1 - \Omega_0) + \Omega_0 a\right]^{-1/2} da$$

LHS is going to be arcsin of something. Let us do three cases.

- **$\Omega_0 = 1$**

$$\text{LHS} = H_0^{-1} \int_a^1 x^{-1/2} dx$$
$$= 2 H_0^{-1}\left(1 - a^{1/2}\right)$$
$$\text{RHS} = r_0 \arcsin (r/r_0) \to r \text{ since } r_0 \to \infty$$

Result

$$r(a) = 2\, H_0^{-1}\left(1 - a^{1/2}\right)$$

- $\Omega_0 = 0$

$r_0 = i\, H_0^{-1}$

$\text{RHS} = i\, H_0^{-1} \arcsin\left[r/\left(i\, H_0^{-1}\right)\right]$

$\text{LHS} = H_0^{-1} \int_a^1 a^{-1}\, d\, a$

$= -H_0^{-1} \log a$

Collect LHS=RHS to get

$\iota \log a = \arcsin\left[r/\left(i\, H_0^{-1}\right)\right]$

$\frac{1}{2\, i}\left(e^{i^2 \log a} - e^{-i^2 \log a}\right) = \frac{r}{i\, H_0^{-1}}$

$$r(a) = \frac{1}{2}\, H_0^{-1}\left(a^{-1} - a\right)$$

- $\Omega_0 = 2$

$r_0 = H_0^{-1}$

$$r(a) = H_0^{-1}(1 - a)$$

Angle subtended by a ruler

We observe a ruler at a great distance. What is the angle between the ends of the ruler? (This observation has been made by WMAP: The ruler was c×(the age of the universe at recombination).)

Consider two events:
Event A: One end of the ruler emits some light at time t_1. The comoving coordinate is r_1.
Event B: Other end of the ruler emits some light at time t_1.

The path for the two light pulses: θ is constant and r changes from r_1 to 0.

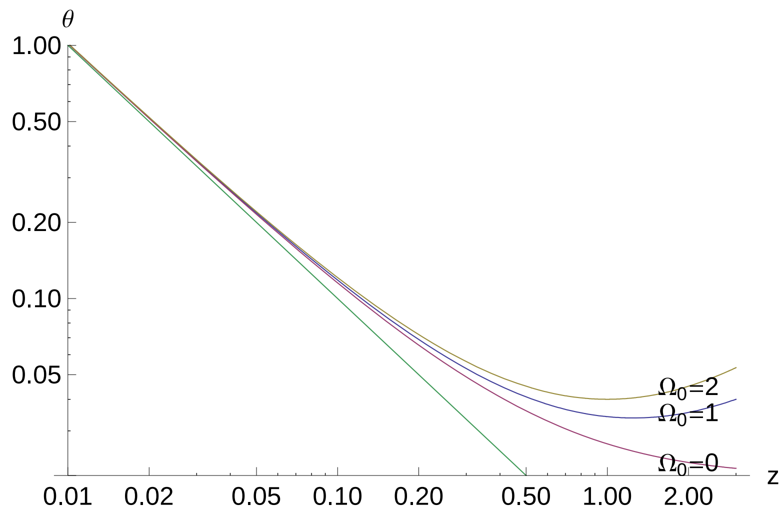
$$d s^2 = -d t^2 + a(t)^2 \left[d r^2 / \left(1 - (r/r_0)^2\right) + r^2\left(d \theta^2 + \sin^2 \theta\, d \phi^2\right)\right]$$

The distance between the two events is L, the length of the ruler.

$L = a(t_1)\, r_1\, \theta$

Drop the subscripts, since it is clear a and r refer to the epoch and location of the ruler emitting the pulse of light.

$\theta = \frac{L}{a\, r(a)}$



Caption: Angle subtended by a ruler of length 0.01 H_0^{-1} The green line is $\theta = L/(z\, H_0^{-1})$.

Q: Why do all lines match $\theta = L/(z\, H_0)$ for $z \ll 1$?
Q: Why is there a minimum angle for some cases?

- **Plots**

Initialization