Sound waves before recombination

- Outline
  - Is sound carried by radiation or by matter?
  - Sound speed
  - Calculation of the angular scale for a universe dominated by radiation
  - Precise calculation of the angular scale

- Physical conditions at recombination

- At recombination, which has the greater mass density, pressureless matter or radiation?

\[
\Omega_{m0} = 0.26 \quad \text{pressureless matter, mostly dark matter, matter that does not interact with light}
\]

\[
\Omega_{b0} = 0.043 \quad \text{baryons, ordinary matter}
\]

\[
\Omega_{r0} = 1.2 \times 10^{-5} \quad \text{radiation}
\]

\[
\rho_b = \rho_{b0} a^{-3}
\]

\[
\rho_r = \rho_{r0} a^{-4}
\]

\[
\rho_m / \rho_r = \rho_{m0} / \rho_{r0} a
\]

At \(a_{eq} = 0.00028 (z = 3600)\), the mass-energy density of baryonic matter and radiation are equal.

Q: At recombination (\(a=0.0009\)), which has greater mass density, pressureless matter or radiation?

We will discuss sound waves, which has to do with radiation and matter.

Q: Does dark matter participate in the sound waves?

Q: At recombination, are electrons pressureless? The energy of a CBR photon is \(2.3 \times 10^{-4} \text{ eV}\).

- Which has greater number density, baryonic matter or radiation?

At the present time, the mass of baryonic matter is 938MeV.

The mass of a photon is

\[
2.73 \text{ K}/(11600 \text{ K/eV}) = 2.3 \times 10^{-4} \text{ eV}
\]

The number density

\[
n_e/n_b = 0.00028 \times 938 \text{ MeV}/2.3 \times 10^{-4} \text{ eV} = 1.1 \times 10^9.
\]

More precisely, because photons have different energies, I need to integrate the Planck number spectrum.
\[ n_r = 0.41 \times 10^9 \text{ photon } m^{-3}(T/2.725 \text{ K})^3 \]
\[ n_b = 0.25 \text{ nucleon } m^{-3}(\Omega_{bh0}/.043) (H_0/72 \text{ km/s/Mpc})^2 \]
\[ n_r/n_b = 1.64 \times 10^9. \]

The number of photons and baryons do not change. As the universe expands, the number of baryons in a coexpanding box does not change. The number of baryons entering must equal the number exiting, because of homogeneity. Same argument is true for photons.

How can you say \( n_r/n_b = 1.64 \times 10^9 \) in a dramatic way?

- **Is sound carried by photons or by matter?**

  The composition of the gas is ordinary matter and photons.

- **Does matter or radiation provide more pressure?**

  Pressure \( P = n p_x v_x \), where \( n \) is number density, \( p \) and \( v \) are momentum and speed.

  For matter,\[
  P = n m v_x^2 = \frac{1}{3} n m v^2.
  \]

  For radiation,\[
  P = \frac{1}{3} n_r E \text{ (from earlier class).}
  \]

  Equipartition: In thermal equilibrium, the average energy of each particle is the same. (More precisely, the energy of each degree of freedom is \( \frac{1}{2} k T \). In QM, degrees of freedom may be frozen with 0 energy.)

  For matter
  \[
  P_m = \frac{2}{3} n m \frac{1}{2} k T = n m k T.
  \]

  For radiation: \( P_r \approx \frac{1}{3} n_r \frac{1}{2} k T \). More accurately,
  \[
  P_r = 0.90 n_r k T.
  \]

  There are \( 10^9 \) photons for every baryon. Therefore the pressure of radiation is \( 10^9 \) times the pressure of matter.

- **Sound speed is** \( \left( \frac{\partial P}{\partial \rho} \right)^{1/2} \)

  The temperature at recombination is 3000K

  There are many photons for every baryon or electron.

  \[ n_r/n_b = 1.64 \times 10^9 \]

  At \( a_{eq} = 0.00028 (z = 3600) \), the mass-energy density of baryonic matter and radiation are equal.

  The speed of sound
  \[ v_s = \left( \frac{\partial P}{\partial \rho} \right)^{1/2} \]

  where the derivative is for adiabatic changes.
Proof: Newton’s 2nd law \( F = ma \) determines the movement of a sound wave. The force is due to an excess pressure. The mass is due to an excess density. Consider a slab of gas between \( x \) and \( x + dx \). Because of the presence of the disturbance, \( x \) moves to \( x + \chi(t, x) \). The \( ma \) term becomes

\[
(m_0 \, dx) \, \frac{\partial \chi}{\partial t}.
\]

The force comes from the difference in pressure. The force term is

\[
- \frac{\partial P}{\partial x} \, dx.
\]

I need to relate pressure to mass density:

\[
P = - \frac{\partial P}{\partial \rho} \frac{\partial \chi}{\partial \rho}.
\]

Collect all; cancel \( m_0 \) and \( dx \):

\[
\frac{\partial \chi}{\partial x} = \frac{\partial P}{\partial \rho} \frac{\partial \chi}{\partial \rho}.
\]

The speed of sound is

\[
v_s = \left( \frac{\partial P}{\partial \rho} \right)^{1/2}.
\]

The derivative is taken with no heat flow, if the wavelength is large compared to the mean-free path.

- **Sound speed for a perfect gas**

For adiabatic changes

\[
P = \text{const} \, \rho^\gamma.
\]

For a monotonic gas, \( \gamma = \frac{5}{3} \).

Take derivative to get

\[
v_s^2 = \gamma \, P / \rho
\]

\[
= \gamma \, PV / (\rho \, V)
\]

\[
= \gamma \, kT / \mu
\]

Equi-partition \( \Rightarrow \frac{3}{2} \, kT = \frac{1}{2} \, m \, v_{avg}^2 \)

Therefore

\[
v_s = \left( \frac{\gamma}{3} \right)^{1/2} \, v_{avg}
\]

The sound speed is approximately the average speed of the gas particles.

- **Values**

---

**Calculating the sound speed**

Consider a box of gas with a fixed number of particles. The box expands or shrinks because of the sound wave.

1. \( d \, U = d \, Q - P \, d \, V = -P \, d \, V \)

   Because there is no heat flow, \( d \, Q = 0 \).

   \( d \, U = -P \, d \, V \).

Recall \( u = a_B \, T^4 \) and \( P = \frac{1}{3} \, a_B \, T^4 \)

\[
d \, (u \, V) = V \, du + u \, d \, V = -P \, d \, V
\]

\[
d \, u = -(u + P) \, d \, V / V
\]

\[
4 \, T^3 \, d \, T = -(T^4 + \frac{1}{3} \, T^4) \, dV / V
\]

\[
3 \, d \, T / T = -d \, V / V
\]
2. \( \frac{d P}{d T} = \frac{4}{3} a_B T^3 \frac{d T}{d T}. \)

3. \( d \rho = d \rho_b + d \rho_r \)
\[ \frac{d \rho_b}{d V} = -\rho_b \frac{d V}{V}, \quad \text{since mass of the baryons in the box (} \rho_b V \text{) is unchanged.} \]
\[ d \rho_b = 3 \rho_b \frac{d T}{T} \]
\[ d \rho_r = 4 a_B T^3 \frac{d T}{d T} \]

4. Gather all:
\[ v_s^2 = \frac{d P}{d \rho} = \left( \frac{4}{3} a_B T^3 \frac{d T}{d T}\right) \left( 3 \rho_b \frac{d T}{T} + 4 a_B T^3 \frac{d T}{d T} \right)^{-1} \]
\[ = \left( 3 + \frac{9}{4} \frac{\rho_b}{\rho_r} \right)^{-1} \]
\[ v_s = \left[ 3 \left( 1 + R \right) \right]^{-1/2}, \]

where
\[ R = \frac{3}{4} \frac{\rho_b}{\rho_r} \]

Q: If \( R \ll 1 \), how fast do sound waves travel?

Q: Why do baryons slow the speed of sound? Recall \( v_s = \left( \frac{d P}{d \rho} \right)^{1/2} \).

Number density of photons:
\[
\text{Integrate}\left[ \frac{x^2}{(e^x - 1), \{x, 0, \infty\}} \right] \]
2 \( \text{Zeta}\left[3\right] \)

Energy density:
\[
\text{Integrate}\left[ \frac{x^3}{(e^x - 1), \{x, 0, \infty\}} \right] \]
\[ \frac{\pi^4}{15} \]

Average energy:
\%
\%

2.70118

\[ \frac{1}{3} \langle E \rangle \text{ is} \]
\%
3

0.900393

**Calculating the horizon**

How far does a sound wave travels from the big bang (\( t=0 \)) to the time of recombination? Let
\[ a_L \quad \text{be the expansion parameter at last scattering (recombination)} \]
\[ a_E \quad \text{be the expansion parameter at epoch when} \ \rho_r = \rho_b. \]
The distance of the horizon is 
\[ d = \int v_s \, dt. \]

\( v_s \, dt \) is how far the sound wave moves. As the wave is moving, the ending point is moving too. Calculation is wrong.

Better posed: What is the comoving coordinate \( r \) of a sound wave that travels from the big bang (\( t=0 \)) to the epoch of recombination?

\[ v_s \, dt = a \, dr \]
\[ r = \int_0^r v_s \, a^{-1} \, dt. \]

Given \( r \), how do you get the distance of the horizon? \( d = a_L \, r \)
\[ d = a_L \int_0^r v_s \, a^{-1} \, dt. \]

The sound speed \( v_s = \frac{\Omega k}{\sqrt{1 + R}} \) depends on \( R = \frac{3}{4} \frac{\rho_k}{\rho_r} = \frac{3}{4} \left( \frac{a}{a_e} \right). \)

- **To gain understanding, consider this simplified case where matter is negligible:** \( R \ll 1 \).

Then
\[ d = a_L \int_0^r v_s \, a^{-1} \, dt \]

Use Friedman’s equation
\[ \left( \frac{da}{d\ln a} \right)^2 = (\Omega k_0 + \Omega_m a^{-1} + \Omega_r a^{-2}) \rightarrow \Omega_0 a^{-2} \]
\[ d = a_L \int_0^r v_s \, a^{-1} \, dt \]
\[ = H_0^{-1} a_L \int_0^r v_s \Omega_0^{-1/2} \int_0^a d a \]
\[ = H_0^{-1} a_L^2 v_s \Omega_0^{-1/2} \]

Apply F’s eqn at \( a_L \)
\[ H(a_L) = H_0 \Omega_0^{1/2} a_L^2 \]
to get the transparent result
\[ d = H^{-1} (a_L) \, v_s \]

Q: Interpret the formula for \( d \).

A dense region produces a sound wave that goes in all directions to cover a length \( 2 \, d \).

The angle subtended is
\[ \theta = \frac{2d}{ra_L} \]
\[ = 2 H^{-1} (a_L) \, v_s / [a_L \, r(a_L)] \]

Q: Interpret the formula for \( \theta \).

- **Results for best cosmological values**

\[ d = a_L \int_0^r v_s (a) \, dt \]

Change \( dt = H^{-1} a^{-1} \, da \), and integrate to get (Weinberg 2008, Cosmology, p. 145)
\[ d = 2 H_0^{-1} a_L^{3/2} (3 R_L \Omega_{m0})^{-1/2} \ln \left[ \left( 1 + R_L \right)^{1/2} + (R_L + R_E)^{1/2} \right] / (1 + \sqrt{R_E}) \]

For \( \Omega_{m0} = 0.26, \Omega_r = 0.043, \)
\[ R_L = 0.62 \]
\[ R_E = 0.21 \]
\[ d = 1.16 H_0^{-1} a_L^{3/2} \]
\[ \theta = \frac{2d}{\pi \langle \alpha \rangle _L} = \frac{1}{48} = 1.2^\circ \]