13 Mar 2012—Equivalence Principle. Einstein’s path to his field equation

15 Mar 2012—Tests of the equivalence principle

- Outline:
    - Proof of Noether’s theorem
    - Experimental tests of the equivalence principle.

Homeowner’s frame and painter’s frame

A painter fell off the roof and feels no gravity. Paint frops are in free fall. Standing on the ground, the homeowner sees that the paint drops accelerate. Find the transformation between the painter’s frame ξ^μ and the homeowner’s frame x^ν.

$$d x^\nu = \frac{\partial x^\nu}{\partial \xi^\mu} \, d \xi^\mu$$

Try this transformation

$$x^0 = \xi^0$$

$$x^1 = \xi^1 - \frac{1}{2} \, g \left(\xi^0\right)^2$$

on a paint particle that is moving at constant speed in the painter’s frame.

$$\xi_{\text{pp}} = \left(\xi^0, \, \xi^1_{\text{pp}} + v \, \xi^0\right)$$

In the homeowner’s frame,

$$x_{\text{pp}} = \left(t, \, \xi^1_{\text{pp}} + v \, t - \frac{1}{2} \, g \, t^2\right)$$

which is accelerated fall.

The nonzero transformation derivatives are

$$\frac{\partial x^0}{\partial \xi^0} = 1$$

$$\frac{\partial x^1}{\partial \xi^1} = 1$$

$$\frac{\partial x^1}{\partial \xi^0} = -g \, \xi^0$$

Momentum conservation

We prove Noether's theorem: If the metric does not change with a translation in the ν-th coordinate, then p_ν is conserved.

http://owpdb.mfo.de/detail?photo_id=9267

Amalie Noether, March 23, 1882 - April 14, 1935.

We will show that

$$\frac{d\,u_\nu}{d\tau} - \frac{1}{2}\, g_{\alpha\beta,\nu}\, u^\alpha\, u^\beta = 0.$$

Then the proof is easy. If the metric does not change with a translation in the ν coordinate, then $g_{\alpha\beta,\nu} = 0$ and $\frac{d\,u_\nu}{d\tau} = 0$.

Proof:

$$\frac{du_\nu}{d\tau} = \frac{d\, g_{\nu\mu}\, u^\mu}{d\tau} = g_{\nu\mu}\, \frac{du^\mu}{d\tau} + u^\mu\, g_{\nu\mu,\sigma}\, \frac{dx^\sigma}{d\tau}$$

Q: Although I am doing some algebra, I want to know what I am doing. What does it mean to compute $\frac{d}{d\tau}\, g_{\nu\mu}$? What is the difference between $\frac{d}{d\tau}\, g_{\nu\mu}$ and $\frac{\partial}{\partial t}\, g_{\mu\nu}$?

$$\begin{aligned}
\frac{du_\nu}{d\tau} &= \frac{d\, g_{\nu\mu}\, u^\mu}{d\tau} = g_{\nu\mu}\, \frac{du^\mu}{d\tau} + u^\mu\, g_{\nu\mu,\sigma}\, \frac{dx^\sigma}{d\tau} \\
&= -g_{\nu\mu}\, \Gamma^\mu_{\alpha\beta}\, u^\alpha\, u^\beta + g_{\nu\mu,\sigma}\, u^\mu\, u^\sigma \\
&= -\frac{1}{2}\, g_{\nu\mu}\, g^{\mu\gamma}\left(g_{\alpha\gamma,\beta} + g_{\gamma\beta,\alpha} - g_{\beta\alpha,\gamma}\right) u^\alpha\, u^\beta + g_{\nu\mu,\sigma}\, u^\mu\, u^\sigma \\
&= -\frac{1}{2}\left(g_{\alpha\nu,\beta} + g_{\nu\beta,\alpha} - g_{\beta\alpha,\nu}\right) u^\alpha\, u^\beta + g_{\nu\mu,\sigma}\, u^\mu\, u^\sigma
\end{aligned}$$

Since α, β, μ, and σ are indices that are summed,

$$\begin{aligned}
\frac{du_\nu}{d\tau} &= -\frac{1}{2}\left(g_{\alpha\nu,\beta} - g_{\nu\beta,\alpha} - g_{\beta\alpha,\nu}\right) u^\alpha\, u^\beta \\
&= \frac{1}{2}\, g_{\beta\alpha,\nu}\, u^\alpha\, u^\beta
\end{aligned}$$

Inertial and gravitational mass

Mass appears in two contexts:

In Newton's 2nd law,

$$F = m_i\, a$$

and in Newton's law of gravity,

$$\begin{aligned}
F_{\text{grav}} &= \frac{G\, M_{\text{earth}}}{r^2}\, m_g \\
&= m_g\, g
\end{aligned}$$

Call these inertial and gravitational masses. That these are the same is the foundation of Einstein's gravity.

Einstein:

"*...for an observer falling freely from the roof of a house there exists—at least in his immediate surroundings— no gravitational field...*

Because of this idea, the uncommonly peculiar experimental law that in the gravitational field all bodies fall with the same acceleration attained at once a deep physical meaning. Namely, if there were to exist just one single object that falls in the gravitational field in a way different from all others, then with its help the observer could realize that he is in a gravitational field and is falling in it. If such an object does not exist, however—as experience has shown with great accuracy—then the observer lacks any objective means of perceiving himself as falling in a gravitational field. Rather he has the right to consider his state as one of rest and his environment as field-free relative to gravitation."

Test of equivalence with a pendulum

Galileo on an test of the equivalence :
"I took two balls, one of lead and one of cork, the former being more than a hundred times as heavy as the latter, and suspended them from two equal thin strings... Pulling each ball aside from the vertical, I released them at the same instant, and they... passed thru vertical and returned along the same path. This free oscillation, repeated more than a hundred times, showed clearly that the heavy body kept time with the light body so well that neither in a hundred oscillations, nor in a thousand, will the former anticipate the latter even by an instant, so perfectly do they keep in step." Galileo, 1638, from Ohanian & Ruffini, Gravitaion and Spacetime, 1994, p. 25.

Newton: I tried the thing in gold, silver, lead, glass, sand, common salt, wood, water, and wheat. I provded two equal wooden boxes. I filled the one with wood, and suspended an equal weight of gold (as exactly as I could) in the center of oscillation of the other. The boxes, hung by equal threads of 11 feet, made a couple of pendulums perfectly equal in weight and figure, and equally exposed to the resistance of air. Placing one by the other, I observed them to play together forwards and backwards fo a long while with equal vibrations... And by these experiments, in bodies of the same weight, one could have discovered a difference of matter less than a thousandth part of the whole." Newton 1686, from Ohanian & Ruffini, Gravitaion and Spacetime, 1994, p. 25.

The gravitational force is $m_g\, g \sin\theta$.

Newton's second law:

$$m_i\, L\, \ddot{\theta} = m_g\, g \sin\theta$$

The frequency is

$$\omega = \left(\frac{m_g}{m_i}\,\frac{g}{L}\right)^{1/2}$$

Summary of tests of equivalence principle

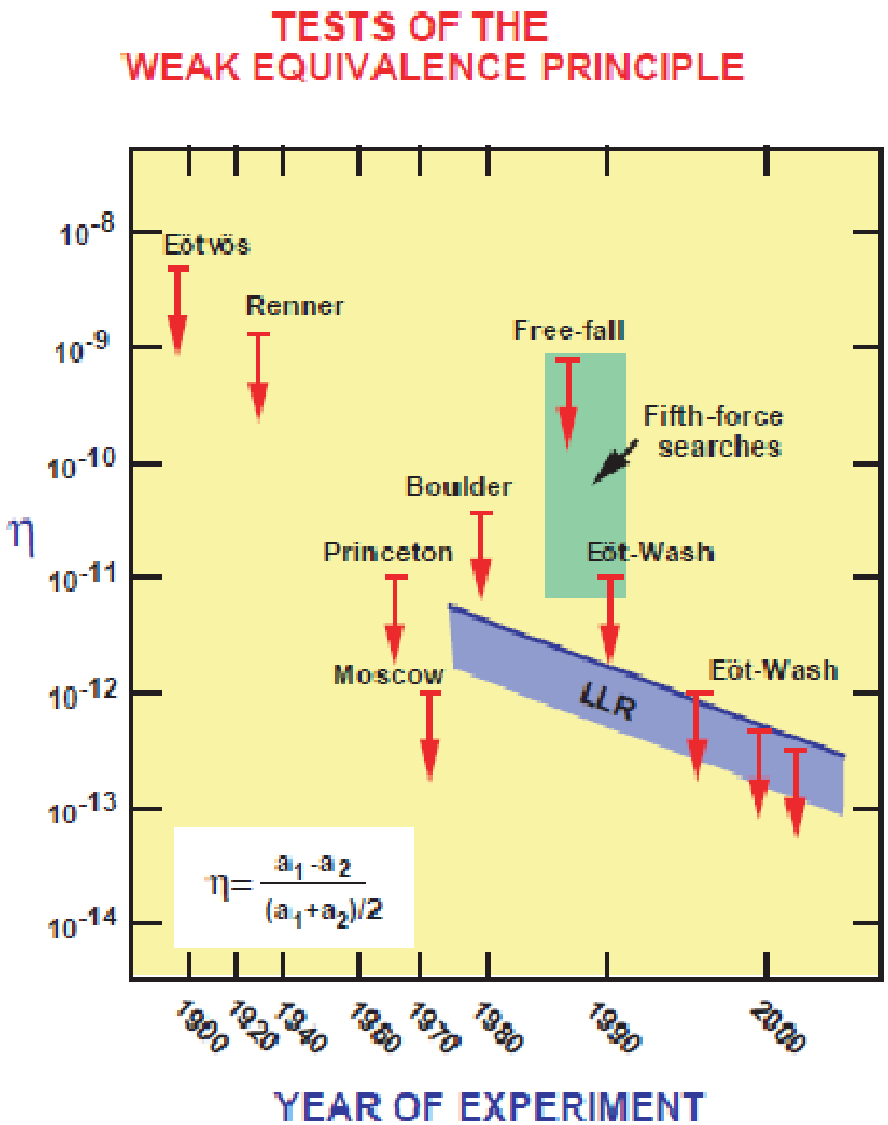
Because of $E = m\, c^2$, the mass is composed parts: mass of the constituents, electromagnetic energy, weak energy, and gravitational energy. Two different materials have a different mixture of each. It is possible that these parts contribute differently to the inertial and gravitational masses. The Eötvös parameter

$$\eta = \left| a_1 - a_2 \right| / \overline{a}$$

characterizes the difference in the acceleration of two materials.

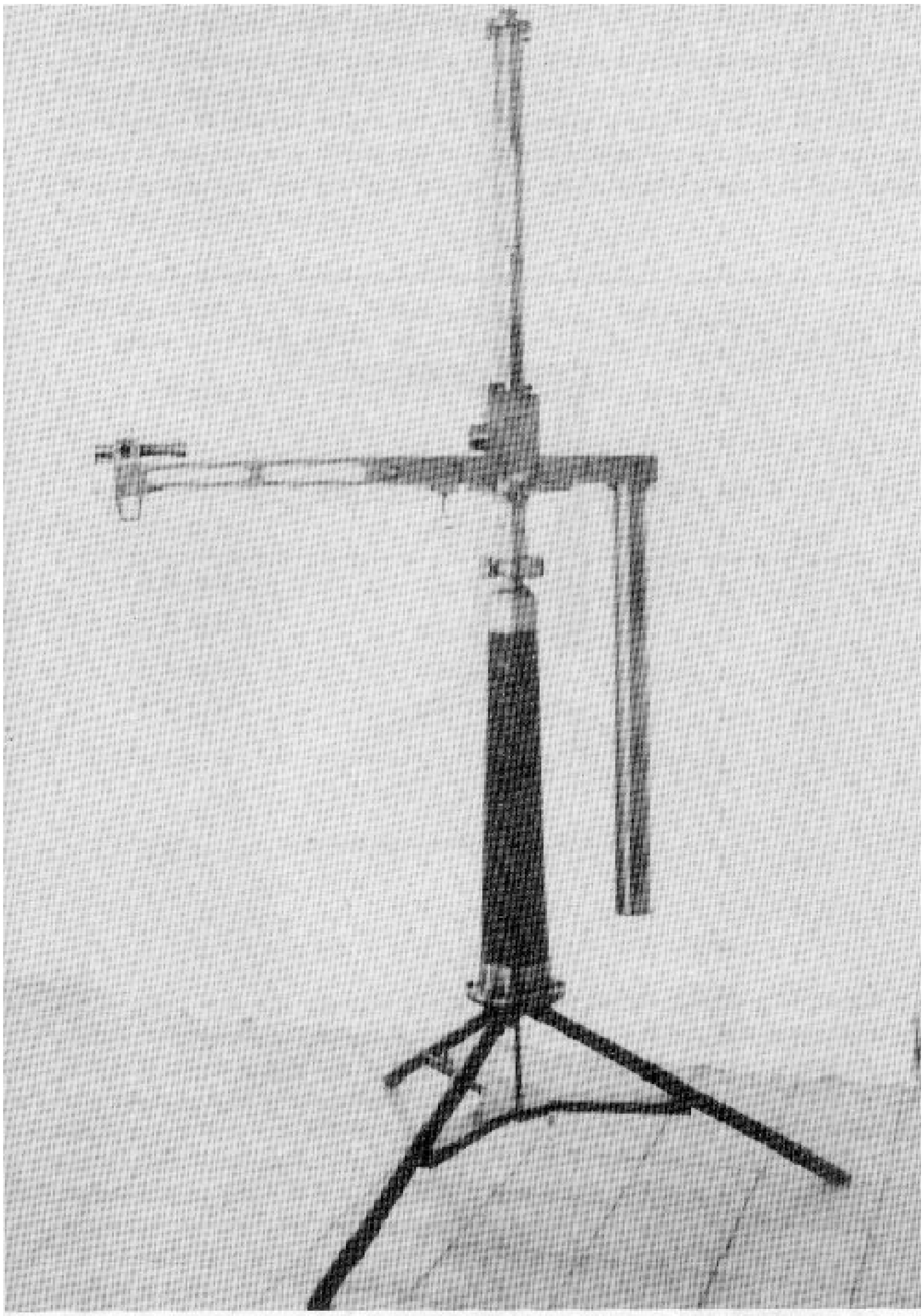
The weak equivalence principle: In a small region, gravity and acceleration are indistinguishable.
Stong equivalence principle: In all freely falling and non rotating frame, the laws of physics are the same.

Q: Eötvös tested wood against metal. Was he able to determine whether chemical energy contributes to the mass?



Clifford M. Will, "The Confrontation between General Relativity and Experiment",
Living Rev. Relativity, 9, (2006), 3, cited: 15 Mar 2010, http://www.livingreviews.org/lrr-2006-3

Eötvös' experiment



Two masses m_1 and m_2 are suspended on a balance.
In the vertical direction, the acceleration of gravity g and centripetal acceleration a_z both act. The apparatus tilts so that it is balanced:

$$l_1(m_{g1}\, g - m_{i1}\, a_z) = l_2(m_{g2}\, g - m_{i2}\, a_z)$$

In the horizontal direction, the centripetal acceleration a_x causes a torque

$$\tau = (l_1\, m_{i1} - l_2\, m_{i2})\, a_x$$

The first equation determined l_2. Then

$$\tau = a_x\, l_1\, m_{i1}\left[1 - \left(\frac{m_{g1}}{m_{i1}}\, g - a_z\right)\left(\frac{m_{g2}}{m_{i2}}\, g - a_z\right)^{-1}\right]$$
$$= l_1\, a_x\, m_{g1}\left(\frac{m_{i1}}{m_{g1}} - \frac{m_{i2}}{m_{g2}}\right)\left(1 - \frac{a_z}{g}\, \frac{m_{i2}}{m_{g2}}\right)^{-1}$$
$$\approx l_1\, a_x\, m_{g1}\left(\frac{m_{i1}}{m_{g1}} - \frac{m_{i2}}{m_{g2}}\right)$$

What are difficulties?

Eötvös rotated the balance 180° to eliminate assymetries within the balance.

Q: Estimate a_x.

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6000*^3 / (24 × 3600.)^2 Meter / Second^2
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$$\frac{0.000803755\ \text{Meter}}{\text{Second}^2}$$

Eötvös determined $\frac{m_{i1}}{m_{g1}}$ was the same for wood and platinum to 5×10^{-9}. The upper limit to the accereration was

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5 × 10^-9 %
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$$\frac{4.01878\times10^{-12}\ \text{Meter}}{\text{Second}^2}$$

Q: Does the observer cause an effect? An 100-kg Baron von Eötvös at 1m produces an acceleration of

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<< PhysicalConstants`
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GravitationalConstant 100 Kilogram / Meter^2
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$$\frac{6.67428\times10^{-9}\ \text{Newton}}{\text{Kilogram}}$$

Q: Does the signal change with time of day?

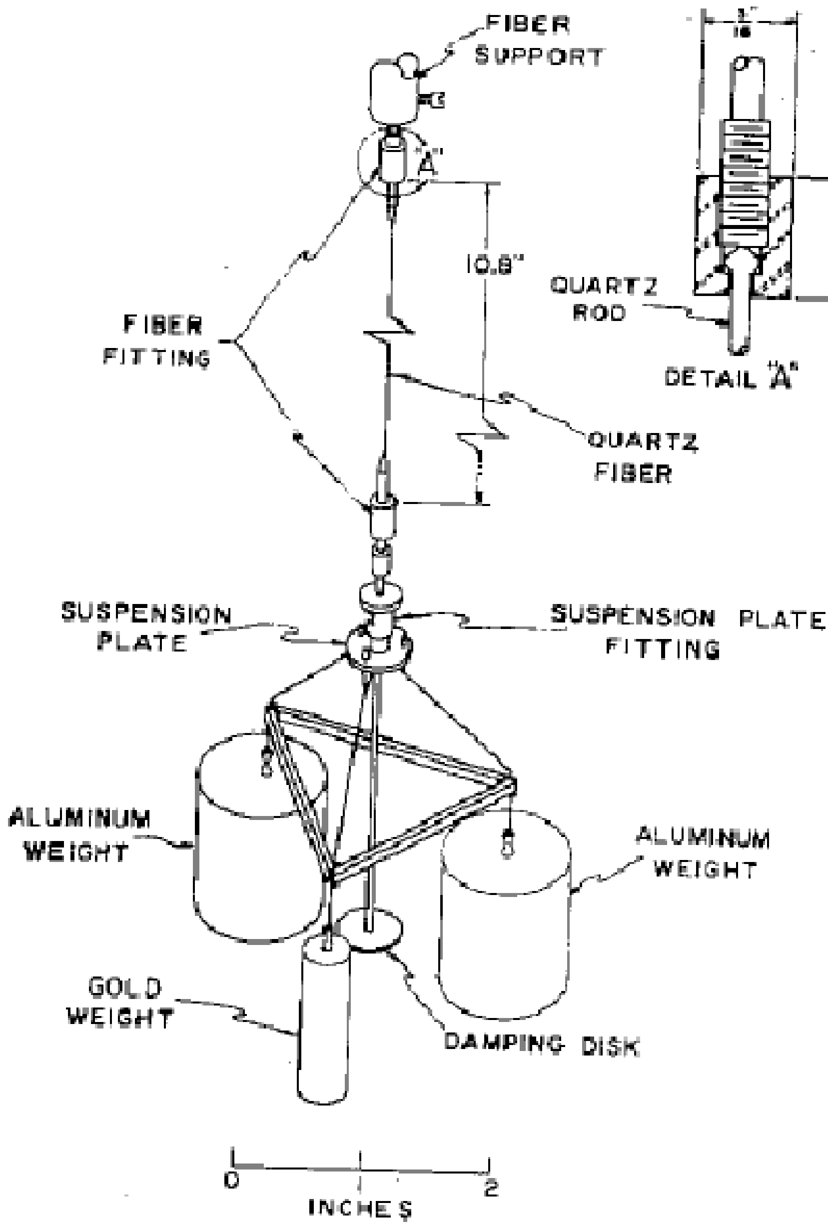
Q: Does local geology cause an effect?

Dicke's apparatus

Measure the pull of the sun and the centripetal acceleration of the orbit around the sun on aluminum and gold.

1. The signal is not static. the sun changes direction every 24 hours.
2. The acceleration is larger: $0.6\ \text{cm}/\text{s}^2$ vs $0.08\ \text{cm}/\text{s}^2$ for the spin of the earth.

Pictures and plots are from Dicke, R., 1970, *Gravitation and the Ubiverse*, American Philosophical Society, Philadelphia.



FIBER SUPPORT
"A"
10.8"
FIBER FITTING
QUARTZ FIBER
QUARTZ ROD
DETAIL "A"
SUSPENSION PLATE
SUSPENSION PLATE FITTING
ALUMINUM WEIGHT
ALUMINUM WEIGHT
GOLD WEIGHT
DAMPING DISK
0
2
INCHES

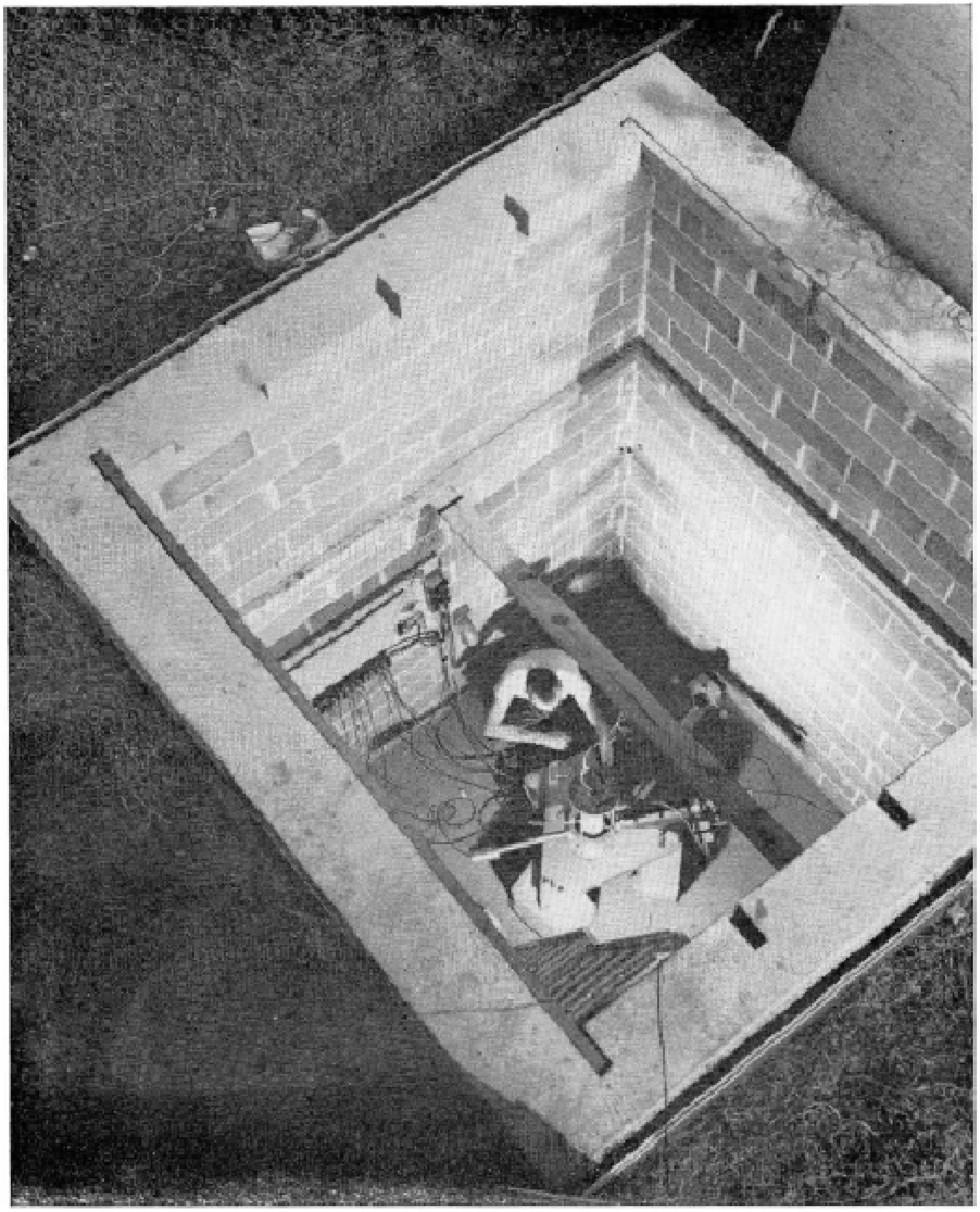


FIG. 5. The instrument well housing the torsion balance. In operation the well is un-manned and is capped with a 3-foot insulated plug carrying embedded electric blankets.

Measuring the torque on the balance

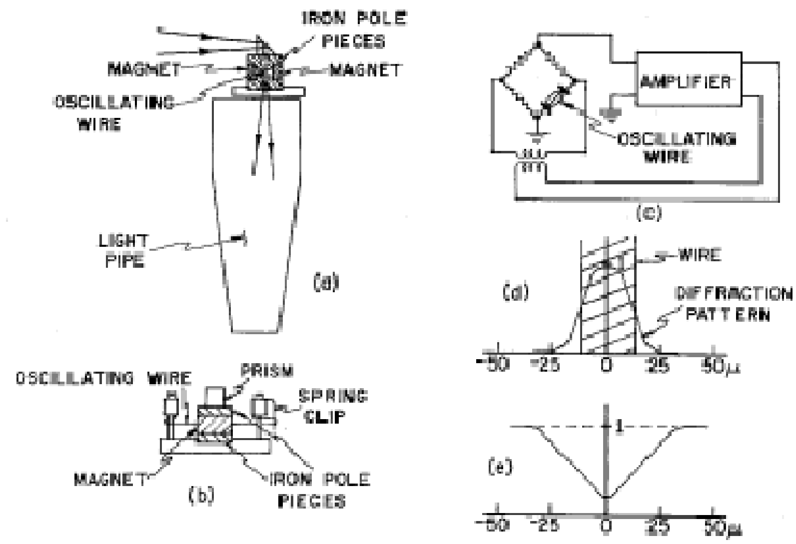


FIG. 8. The oscillating-wire detector of small rotations.

Measurement and zeroing the angle of the torsion balance. Light is reflected off a mirror on the balance and focused on a wire oscillating at frequency f_0. A torque is applied to the balance to center the light on the wire. An error of 10^{-9}radian can be detected.

Q: If the light is off center, what is the frequency of the signal? If the light is centered, what is the frequency of the signal?

Gas pressure

Requirement: Measure an acceleration $10^{-11}\, g_{\text{sun}} = 10^{-11}\, 0.6\,\text{cm}/\text{s}^2$

The vacuum pressure is 10^{-11} atm $= 10^{-6}\,\text{N}/\text{m}^2$.

What temperature difference causes the pressure difference to make an acceleration of this size?

$$P\,A\,\frac{\delta \text{T}}{T}\,\frac{1}{m} = 10^{-11}\, g_{\text{sun}}$$

$$\frac{\delta \text{T}}{T} = 10^{-11}\, 0.006\,\text{m}/\text{s}^2\; 1\,\text{kg}/\left(10^{-6}\,\text{N}/\text{m}^2\right)/\left(0.03\times 0.1\,\text{m}^2\right) = 0.00002$$

$$\delta\text{T} = 0.006\,\text{C}$$

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10^-11 .6*^-2 / (10^-6 .03 × .1)
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0.00002
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```
300 %
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0.006
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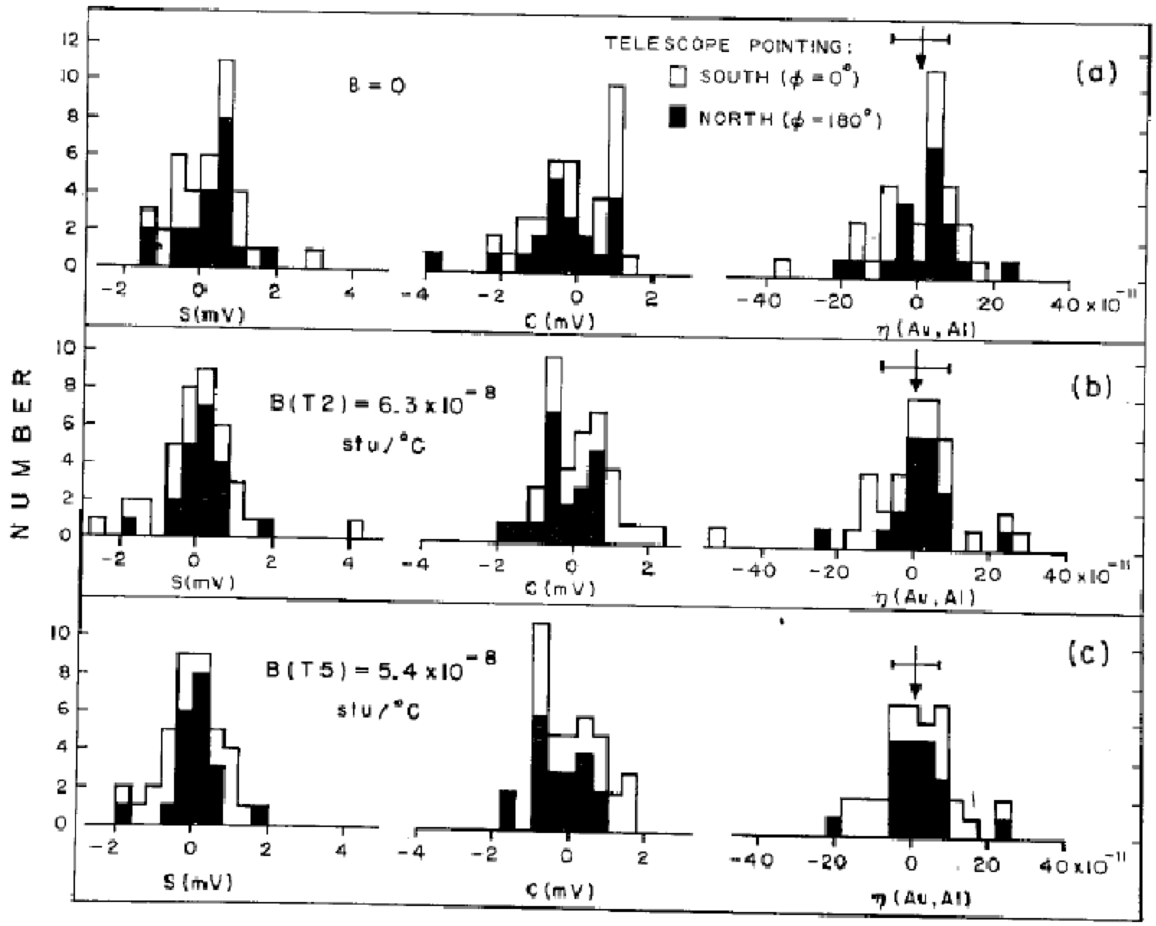
Results



Results: Histograms of one-day averages. Left and center columns: sine and cosine components of the torque in mV. Right column: Eötvös η. Top row: raw data. Center: Correlation with temperature sensor T2 removed. Bottom row: Correlation with temperature sensor T5 removed.

Temperature does affect the measurements.
With the temperature correlation removed, the average of η is consistent with 0.

$$\eta = (1.32 \pm 1.04) \times 10^{-11}$$

20 Mar 2012—Tensors

22 Mar 2012—Parallel transport of a vector. Riemann-Christoffel curvature tensor. Bianchi Identity

27 Mar 2012—Stress-energy tensor. Conservation of energy and momentum. Einstein’s discovery of the field equation