13 Mar 2012—Equivalence Principle. Einstein's path to his field equation

15 Mar 2012—Tests of the equivalence principle

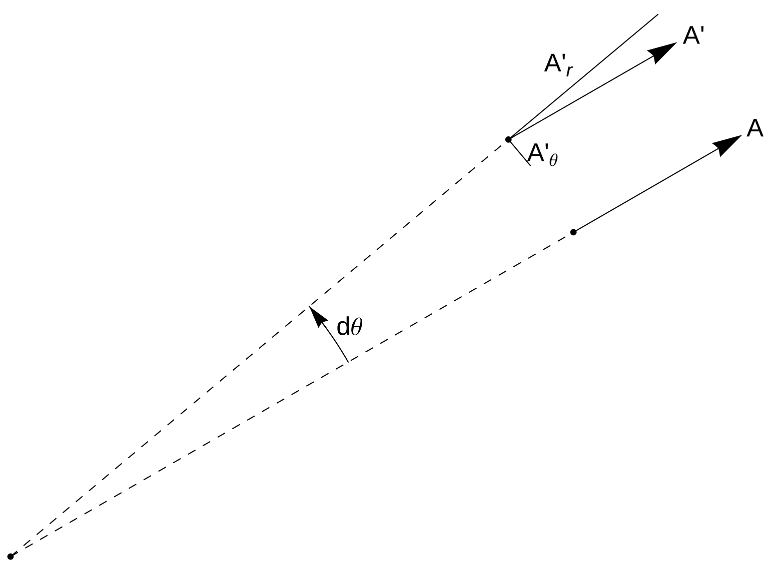
20 Mar 2012—General covariance. Math. Covariant derivative

22 Mar 2012—Riemann-Christoffel curvature tensor.

- Outline
    - Finish covariant derivatives
    - Riemann-Christoffel curvature tensor

Covariant derivative of a contravariant vector

How do you take derivatives of tensors?
Requirements
1) The derivative of a tensor must be a tensor
2) The derivative must measure a physical quantity and not merely a quirk of the coordinate system.



Example of a quirk of the coordinate system: A constant vector field A^μ in two dimensions with polar coordinates. Even though A is constant, $\frac{\partial A^r}{\partial \theta}$ is not zero.

Mathematical problem: The derivative means to compare $A(x + dx)$ and $A(x)$.

Rather than rethink this, use what we did for the equation of motion.

We already found that the equation of motion is

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma = 0.$$

The terms $\frac{du^\alpha}{d\tau}$ and $\Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma$ are not tensors. Proof: $\Gamma^\alpha{}_{\beta\gamma}$ is zero in a gravity-free frame. If it were a tensor, it would be zero in all frames.

We derived the equation of motion by differentiating the 4-velocity.
Rewrite

$$\frac{du^\alpha}{d\tau} = \frac{dx^\beta}{d\tau}\frac{\partial u^\alpha}{\partial x^\beta} = u^\beta \frac{\partial u^\alpha}{\partial x^\beta}$$

and insert to get

$$u^\beta\left(\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma}\, u^\gamma\right) = 0.$$

Contraction is a tensor operation. Therefore $\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma}\, u^\gamma$ is a tensor.

For any contravariant vector A^α,

$$\boxed{\nabla_\beta A^\alpha = \frac{\partial A^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma}\, A^\gamma}$$

is a tensor. This is called the covariant derivative. We have succeeded in defining a “good” derivative.

Another notation:

$$\boxed{A^\alpha{}_{;\beta} = A^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\gamma}\, A^\gamma}$$

Is $A^\alpha{}_{;\beta} \equiv \nabla_\beta A^\alpha$ covariant or contravariant in the index β?

Example: For 2-dimensional polar coordinates, the metric is

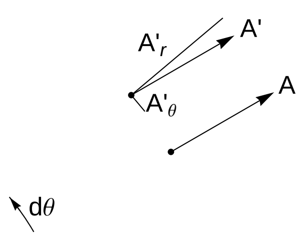
$$ds^2 = dr^2 + r^2\, d\theta^2$$

The non-zero Christoffel symbols are (8.17)

$$\Gamma^r_{\theta\theta} = -r$$
$$\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = 1/r.$$

$$A^r{}_{;r} = A^r{}_{,r}$$
$$A^r{}_{;\theta} = A^r{}_{,\theta} - r\, A^\theta$$
$$A^\theta{}_{;r} = A^\theta{}_{,r} + 1/r\, A^\theta$$
$$A^\theta{}_{;\theta} = A^\theta{}_{,\theta} + 1/r\, A^r$$

The covariant derivative of the r component in the r direction is the regular derivative. If a vector field is constant, then $A^r{}_{;r} = 0$.
The covariant derivative of the r component in the θ direction is the regular derivative plus another term. Even if a vector field is constant, $A^r{}_{;\theta} \neq 0$. The Γ term accounts for the change in the coordinates.



Q: Which of these four terms does the figure illustrate?

The idea of a covariant derivative of a vector field A in the direction a. Is this a good definition?

$$\nabla_a A^\alpha = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[A(x + \epsilon\, a) - A(x)\right] \text{ ???}$$

However, the components of $A(x+\epsilon a)$ may be different even if the vector is the same, because the coordinates are changing. We must move $A(x+\epsilon a)$ back to x before comparing. Moving is called parallel transporting. This is what the Γ term does.

$$\nabla_a A^\alpha = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \{\text{parallel transport}[A(x+\epsilon a)] - A(x)\}$$

Q: Simplicio: Covariant derivatives are irrelavant. I want to know about gravity. In what way is Simplicio mistaken?

- **Plot**

How to measure curvature

In what object is gravity encoded? What does the Equivalence Principle say? Gravity is encoded in a general coordinate transformation.

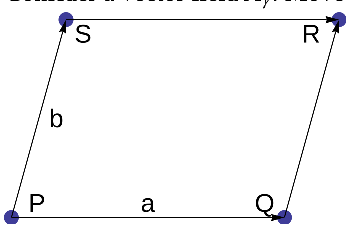
Q: Can you measure curvature by looking at a point?

- **Answer**

-

Q: How to detect curvature of the Earth's surface.
Carry a vector, which points east, from the north pole to the equator.

Consider a vector field A_γ. Move from point P to Q to R. Move from P to S to R. Compare.

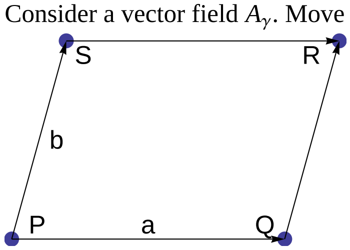


The change in A in going from P to Q is

$$\mathrm{dA}_{\gamma\,\mathrm{PQ}} = \left(\frac{\partial A_\gamma}{\partial x^\alpha}\right) a^\alpha$$

Q: Why is this not a tensor equation?

Consider a vector field A_γ. Move from point P to Q to R. Move from P to S to R. Compare.



The change in A_γ in going from P to Q is

$$d\,A_{\gamma\,\mathrm{PQ}} = \left(\frac{\partial A_\gamma}{\partial x^\alpha}\right) a^\alpha$$

Why is this not a tensor equation?

- **Answer**

-

$$\nabla_\alpha A_\gamma = \frac{\partial A_\gamma}{\partial x^\alpha} - \Gamma^\sigma{}_{\gamma\alpha} A_\sigma$$

This is a tensor equation:

$$d\, A_{\gamma\, \mathrm{PQ}} = \nabla_\alpha A_\gamma\, a^\alpha$$

The change in A in going P→Q→R is

$$d\, A\gamma_{\mathrm{PQR}} = \nabla_\beta (\nabla_\alpha A_\gamma)\, a^\alpha\, b^\beta$$

The change in A in going P→S→R is

$$d\, A_{\gamma\, \mathrm{PSR}} = \nabla_\alpha (\nabla_\beta A_\gamma)\, a^\alpha\, b^\beta$$

The change in a round trip P→Q→R→S→P is

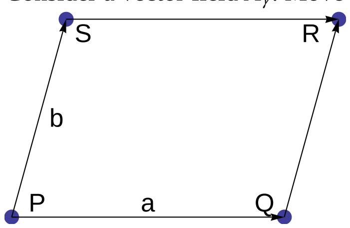
$$d\, A_{\gamma\, \mathrm{PQR}} - d\, A_{\gamma\, \mathrm{PSR}} = [\nabla_\beta (\nabla_\alpha A_\gamma) - \nabla_\alpha (\nabla_\beta A_\gamma)]\, a^\alpha\, b^\beta$$

Q: In MA1, I learned that $\frac{\partial^2}{\partial x\, \partial y} = \frac{\partial^2}{\partial y\, \partial x}$. Why doesn't the quantity in brackets [] = 0?

- **Fig**

How to measure curvature

Consider a vector field A_γ. Move from point P to Q to R. Move from P to S to R. Compare.



The change in A_γ in going from P to Q is

$$d\, A_{\gamma\, \mathrm{PQ}} = \left(\frac{\partial A_\gamma}{\partial x^\alpha}\right) a^\alpha$$

Q: Why is this not a tensor equation? The derivative of a vector field is not a tensor. Use the covarient derivative

$$\nabla_\alpha A_\gamma = \frac{\partial A_\gamma}{\partial x^\alpha} - \Gamma^\sigma{}_{\gamma\alpha} A_\sigma$$

The change in a round trip P→Q→R→S→P is

$$d\, A_{\gamma\, \mathrm{PQR}} - d\, A_{\gamma\, \mathrm{PSR}} = [\nabla_\beta (\nabla_\alpha A_\gamma) - \nabla_\alpha (\nabla_\beta A_\gamma)]\, a^\alpha\, b^\beta$$

In MA1, I learned that $\frac{\partial^2}{\partial x\, \partial y} = \frac{\partial^2}{\partial y\, \partial x}$. Why doesn't the quantity in brackets [] = 0? The parts involving partial derivatives of the vector A_γ is 0. The remaining parts involve the Christoffel symbol times A. Therefore, the nonzero part can be written as

$$d\, A_{\gamma\, \mathrm{PQR}} - d\, A_{\gamma\, \mathrm{PSR}} = -A_\sigma\, R^\sigma{}_{\gamma\alpha\beta}\, a^\alpha\, b^\beta.$$

What does this say?

In a round trip, a vector field A_γ changes by the contraction of what?

The parts involving partial derivatives of the vector A_γ is 0. The remaining parts involve the Christoffel symbol times A. Therefore, the nonzero part can be written as

$$\boxed{d\, A_{\gamma\, \mathrm{PQR}} - d\, A_{\gamma\, \mathrm{PSR}} = -A_\sigma\, R^\sigma{}_{\gamma\alpha\beta}\, a^\alpha\, b^\beta}$$

Q: What does this equation say?

- **Ans**

■

The tensor $R^{\sigma}{}_{\gamma\alpha\beta}$is called the Riemann-Cristoffel curvature tensor.

Q: If I swap α and β, is R the same? $R^{\sigma}{}_{\gamma\alpha\beta} = R^{\sigma}{}_{\gamma\beta\alpha}$? What are the last two indices for?

Calculating $R^{\sigma}{}_{\gamma\alpha\beta}$:

$$\nabla_{\beta}\left(\nabla_{\alpha} A_{\gamma}\right) = \nabla_{\beta}\left(\frac{\partial A_{\gamma}}{\partial x^{\alpha}} - \Gamma^{\sigma}{}_{\gamma\alpha} A_{\sigma}\right)$$

$$= \frac{\partial^2 A_{\gamma}}{\partial x^{\beta}\, \partial x^{\alpha}} - A_{\sigma} \frac{\partial}{\partial x^{\beta}} \Gamma^{\sigma}{}_{\gamma\alpha} - \Gamma^{\sigma}{}_{\gamma\alpha} \frac{\partial A_{\sigma}}{\partial x^{\beta}} - \Gamma^{\sigma}{}_{\gamma\beta} \frac{\partial A_{\gamma}}{\partial x^{\alpha}} + \Gamma^{\sigma}{}_{\rho\alpha} \Gamma^{\rho}{}_{\gamma\beta} A_{\sigma}$$

We can ignore the partial derivatives of A, because in the end only the terms in A survive.
It is possible to show that

$$R^{\sigma}{}_{\gamma\alpha\beta} = \frac{\partial}{\partial x^{\alpha}} \Gamma^{\sigma}{}_{\gamma\beta} - \frac{\partial}{\partial x^{\beta}} \Gamma^{\sigma}{}_{\gamma\alpha} + \Gamma^{\sigma}{}_{\alpha\epsilon} \Gamma^{\epsilon}{}_{\gamma\beta} - \Gamma^{\sigma}{}_{\beta\epsilon} \Gamma^{\epsilon}{}_{\gamma\alpha}$$