3 Apr 2012—Derivation of the Schwartzschild and Robertson-Walker metrics

- Outline
  - Derivation of the Schwarzschild metric
  - Derivation of Robertson-Walker metric
  - Friedmann's equation (Thurs)

The Schwarzschild metric

Schwarzschild's formulation of the problem

What is the metric outside a spherically symmetric, static star? Conditions:

1. The metric does not change with time.

2. The metric is spherically symmetric.

3. The metric must be the same as Newton's gravity far from the star.

Recall (13 Mar) we found in the Newtonian limit that

$$g_{\mu\nu} = \begin{pmatrix} -(1+2\phi) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Here $\phi = -\mathrm{GM}\big/\left(r\,c^2\right) = -M/r$ is Newton's gravitational potential. Note the Newtonian limit is accurate to first order in ϕ for the time-time term but not the space-space term because the equation of motion picks out $\Gamma^{\alpha}{}_{00}$. In Newton's laws, space is not curved.

4. Assume the metric is

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

Later we will show that this assumption is not restrictive.

5. What is E's field equation in the space outside the star?

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,R = -8\,\pi\,G\,T_{\mu\nu}$$

$G_{\mu\nu}$ is called Einstein's curvature tensor.

Contract:

$$g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,R\right) = -8\,\pi\,G\,g^{\mu\nu}\,T_{\mu\nu}$$

Q: Prove $g^{\mu\nu}\,g_{\mu\nu} = 4$?

$$R - \frac{1}{2}\,4\,R = -8\,\pi\,G\,T^{\alpha}{}_{\alpha}.$$

Therefore

$$R_{\mu\nu} = -8\,\pi\,G\left(T_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,T^{\alpha}{}_{\alpha}\right)$$

In the space outside a star, the stress-energy tensor, which is

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

is zero. Therefore

$$\boxed{R_{\mu\nu} = 0}.$$

Calculation in outline

The metric is

$$g_{\mu\nu} = \begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Calculate the Christoffel symbols

$$\Gamma^{\sigma}{}_{\lambda\mu} = \frac{1}{2}\,g^{\nu\sigma}\left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)$$

Calculate the Ricci tensor

$$R_{\mu\kappa} = \frac{\partial}{\partial x^{\kappa}}\,\Gamma^{\lambda}{}_{\mu\lambda} - \frac{\partial}{\partial x^{\lambda}}\,\Gamma^{\lambda}{}_{\mu\kappa} + \Gamma^{\eta}{}_{\mu\lambda}\,\Gamma^{\lambda}{}_{\kappa\eta} - \Gamma^{\eta}{}_{\mu\kappa}\,\Gamma^{\lambda}{}_{\lambda\eta}$$

The condition $R_{\mu\nu} = 0$ imposes constraints on $A(r)$ and $B(r)$.

Calculation in detail

The metric is

$$g_{\mu\nu} = \begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Christoffel symbols

$$\Gamma^{\sigma}{}_{\lambda\mu} = \frac{1}{2}\,g^{\nu\sigma}\left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)$$

We show that $\Gamma^{r}{}_{rr} = \frac{A'}{2\,A}$

$$\Gamma^{r}{}_{rr} = \frac{1}{2}\,g^{\nu r}(g_{r\nu,r} + g_{r\nu,r} - g_{\nu r,r})$$

$$\Gamma^{r}{}_{rr} = \frac{1}{2}\,A^{-1}\,A' + \frac{1}{2}\,A^{-1}\,A' - \frac{1}{2}\,A^{-1}\,A'$$

$$= \frac{1}{2}\,A^{-1}\,A'$$

Other terms:

$$\Gamma^{r}{}_{\theta\theta} = -\frac{r}{A} \qquad \Gamma^{r}{}_{\phi\phi} = -\frac{r\sin^2\theta}{A} \quad \Gamma^{r}{}_{tt} = \frac{1}{2}\,\frac{B'}{A}$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{1}{r} \qquad \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{r} \qquad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta$$

$$\Gamma^{t}{}_{tr} = \Gamma^{t}{}_{rt} = B'/(2\,B)$$

See Hartle, pp. 546–547 for the Christoffel symbols of a spherically symmetric metric. In Hartle,

$$A(r) = e^{\lambda(r,t)}$$

$$B\,(r) = e^{\nu(r,t)}$$

Of course, here $\frac{\partial}{\partial t}\,\nu(r,\,t) = 0$ and $\frac{\partial}{\partial t}\,\lambda(r,\,t) = 0$.

- **Ricci tensor**

The Ricci tensor

$$R_{\mu\kappa} = \frac{\partial}{\partial x^\kappa}\Gamma^\lambda{}_{\mu\lambda} - \frac{\partial}{\partial x^\lambda}\Gamma^\lambda{}_{\mu\kappa} + \Gamma^\eta{}_{\mu\lambda}\Gamma^\lambda{}_{\kappa\eta} - \Gamma^\eta{}_{\mu\kappa}\Gamma^\lambda{}_{\lambda\eta}$$

(Hartle p. 547)

$$R_{rr} = \frac{1}{2}\frac{B''}{B} - \frac{1}{4}\frac{B'}{B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{A'}{A}$$

$$R_{tt} = \frac{-1}{2}\frac{B''}{A} + \frac{1}{4}\frac{B'}{A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\frac{B'}{A}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A}$$

$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$$

The non-diagonal terms are zero.

Get rid of B'':

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0$$

Then

$$d\log A + d\log B = d\log(AB) = 0$$

Therefore

$$A(r)\,B(r) = \text{constant}$$

At $r \to \infty$, the Newtonian limit yields

$$AB \to 1 - 2M/r \to 1$$

Therefore

$$AB = 1.$$

Consider

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A} = 0$$

Since $AB = 1$,

$$\frac{A'}{A} = \frac{-B'}{B}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(2\frac{B'}{B}\right) + \frac{1}{A} = -1 + rB' + B = -1 + \frac{d}{dr}(rB) = 0$$

Integrate to get

$$rB = r + \text{const}$$

or

$$B(r) = 1 + \frac{\text{const}}{r}$$

At ∞,

$$B(r) = 1/A(r) \to 1 + 2\phi = 1 - 2\frac{M}{r}$$

Finally,

$$B(r) = 1 - 2\frac{M}{r}$$

$$A(r) = \left(1 - 2\frac{M}{r}\right)^{-1}.$$

Q: What went into the derivation of Schwartzschild’s metric?

Q: If a star has radial pulsations, does a planet’s orbit change?

Schwarzschild's assumption of the form of the metric

Schwarzschild's assumption of the form of the metric

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

is convenient but not fundamental.

Assume the metric does not depend on time and it depends on space $\vec{x}$ and $\overrightarrow{dx}$ only through the spatial scalars $\vec{x}\cdot\vec{x}$, $\vec{x}\cdot\overrightarrow{dx}$, and $d\vec{x}\cdot d\vec{x}$. The distance^2 depends on all possible quadratic combinations of $(dt,\, d\vec{x})$

$$ds^2 = -F(r)\,dt^2 + 2\,E(r)\,dt\;\vec{x}\cdot d\vec{x} + D(r)\left(\vec{x}\cdot d\vec{x}\right)^2 + C(r)\,dx^2$$

Use spherical coordinates.

$$\left(x^1,\, x^2,\, x^3\right) = r(\sin\theta\cos\phi,\;\sin\theta\sin\phi,\;\cos\theta).$$

Then

$$ds^2 = -F(r)\,dt^2 + 2\,r\,E(r)\,dt\,dr + D(r)\,r^2\,dr^2 + C(r)\,r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

Since the metric is static, we are free to replace t by $t' + f(r)$. The t-t and t-r terms are

$$-F(r)\,(dt' + f'(r)\,dr)^2 + 2\,r\,E(r)\,dr\,(dt' + f'(r)\,dr)$$

If we choose

$$-2\,F(r)\,f'(r) = 2\,r\,E(r),$$

then the $dt\,dr$ term is zero, and

$$ds^2 = -F(r)\,dt^2 + G(r)\,r^2\,dr^2 + C(r)\,r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

where $G(r) = r^2\left[D(r) + E^2(r)/F(r)\right]$

We may change the r coordinate. $r'^2 = r^2\,C(r)$. Then

$$ds^2 = -B(r')\,dt^2 + A(r')\,dr'^2 + r'^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

where

$$B(r') = F(r)$$

$$A(r') = [1 + G(r)/C(r)][1 + r/(2\,C(r))\,C'(r)]^{-2}.$$

Robertson-Walker metric

A 3-d space that is homogeneous and isotropic has a special choice of time. A choice of coordinates is

$$(t,\, r,\, \theta,\, \phi)$$

and the metric is

$$\boxed{ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-(r/r_0)^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right]}$$

$(r,\, \theta,\, \phi)$ is called the comoving coordinate. A galaxy stays at the same position; time changes. r_0^2 can have any value, positive or negative. $a(t)$ is called the expansion parameter.

Friedman's equation is

$$\boxed{\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3}\,G\,\rho = -\frac{1}{r_0^2\,a^2}}$$

We derived Fiedmann's equation except for the constant $-r_0^{-2}$ on the RHS.

Form of the metric

Assumptions:
1) There is a time coordinate that is proper time.
2) At a given time, the space within a small bubble is isotropic.
3) At a given time, the space is homogeneous.

The metric is

$$ds^2 = -dt^2 + A(t, \text{vector } r)\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right)$$

Q: What assumptions have gone into writing a metric of this form?

Since the space is homogeneous,

$$A(t_1, r_1)/A(t_2, r_2)$$

can depend on time and also on the distance between the points. It cannot depend on the location. Therefore

$$ds^2 = -dt^2 + a(t)^2\, B(r)\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right)$$

Q: Have I eliminated the possibility of a curved space by grouping the dr, $d\theta$, and $d\phi$ terms together as $\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right)$, which is the case for flat, spherical coordinates?

By mapping the 3 dimensions onto the surface of a 4-dimensional symmetric space, we can show that the possible metrics are

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-\left(\frac{r}{r_0}\right)^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right),$$

where r_0^2 can be positive or negative.

Derivation of Friedman's equation

The metric is

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-(r/r_0)^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Since the coordinate time is the same as proper time, in average the galaxies are at rest.

Einstein's equation is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

Plan: Calculate the curvature tensors and find conditions on $a(t)$ and r_0 that satisfy E's equations.

Rewrite

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-(r/r_0)^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

as

$$ds^2 = -dt^2 + a(t)^2\left(\tilde{g}_{rr}\, dr^2 + \tilde{g}_{\theta\theta}\, d\theta^2 + \tilde{g}_{\phi\phi}\, d\phi^2\right)$$

where $\tilde{g} = \begin{pmatrix} \left[1-(r/r_0)^2\right]^{-1} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$

Compute the Christoffel symbols

a) Time term

$$\Gamma^0{}_{\alpha\beta} = \frac{1}{2} g^{00}\left(g_{0\alpha,\beta} + g_{0\beta,\alpha} - g_{\alpha\beta,0}\right)$$

$\alpha = 0$:

$$\Gamma^0{}_{0\beta} = \frac{1}{2} g^{00}\left(g_{00,\beta} + g_{0\beta,0} - g_{0\beta,0}\right) = \frac{1}{2} g^{00}\, g_{00,\beta} = 0,$$

because $g_{00} = -1$.

$\alpha = i \neq 0$ and $\beta = j \neq 0$:

$$\Gamma^0{}_{ij} = \frac{1}{2} g^{00}\left(g_{0i,j} + g_{0j,i} - g_{ij,0}\right)$$

The first two terms are zero because ???.

$$\Gamma^0{}_{ij} = -\frac{1}{2}\left(-g_{ij,0}\right) = a(t)\frac{da}{dt}\tilde{g}_{\mathrm{ij}}.$$

$$\boxed{\Gamma^0{}_{ij} = \dot{a}\; a\, \tilde{g}_{ij}}$$

b) Space-time term: $\alpha = 0$ in

$$\Gamma^i{}_{\alpha\beta} = \frac{1}{2} g^{ii}\left(g_{\mathrm{i}\alpha,\beta} + g_{\mathrm{i}\beta,\alpha} - g_{\alpha\beta,i}\right)$$

$$\Gamma^i{}_{0\beta} = \frac{1}{2} g^{ii}\left(g_{\mathrm{i}0,\beta} + g_{\mathrm{i}\beta,0} - g_{0\beta,i}\right)$$

The first term is zero. Second term is zero unless $\beta = i$, in which case it is $\frac{1}{2} g^{\mathrm{ii}}\, 2\, a\, \dot{a}\, \tilde{g}_{\mathrm{ii}} = \frac{\dot{a}}{a}$. The third term is zero, since $g_{00} = -1$.

$$\boxed{\Gamma^i{}_{0\beta} = \Gamma^i{}_{\beta 0} = \frac{\dot{a}}{a}\,\delta^i_\beta}$$

c) Space-space term: $\alpha = i \neq 0$ and $\beta = j \neq 0$ in

$$\Gamma^i{}_{\mathrm{jk}} = \frac{1}{2} g^{\mathrm{ii}}\left(g_{\mathrm{ij},k} + g_{\mathrm{ik},j} - g_{\mathrm{jk},i}\right)$$

involves space coordinates only. Straightforward. Hartle p. 547

$\boxed{\Gamma^r{}_{rr} = \frac{r}{r_0^2}\,\frac{1}{1-(r/r_0)^2}}$ $\boxed{\Gamma^r{}_{\theta\theta} = -r\left[1-(r/r_0)^2\right]}$ $\boxed{\Gamma^r{}_{\phi\phi} = -r\sin^2\theta\left[1-(r/r_0)^2\right]}$ $\boxed{\Gamma^\theta{}_{\mathrm{r}\theta} = \Gamma^\theta{}_{\theta\mathrm{r}} = \Gamma^\phi{}_{\mathrm{r}\phi} = \Gamma^\phi{}_{\phi\mathrm{r}} = 1/r}$

$\boxed{\Gamma^\theta{}_{\phi\phi} = -\cos\theta\sin\theta}$ $\boxed{\Gamma^\phi{}_{\theta\phi} = \cot\theta}$

The other terms are zero.

Form of the Ricci tensor and the source of curvature

What is the form of the space part of the Ricci tensor in the frame in which the matter is at rest? Recall that the universe is isotropic and homogeneous. Take a guess.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

Take the trace

$$R^\mu{}_\mu - \frac{1}{2}\, 4\, R = -8\pi G T^\mu{}_\mu$$

to find that the curvature scalar is related to the trace of the stress-energy tensor:

$$R = 8\pi G T$$

$$\boxed{R_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)}$$

The stress-energy tensor

$$T^{\mu\nu} = (\rho + P)\, u^\mu u^\nu + g^{\mu\nu} P$$

For the frame in which the galaxies are at rest, $u^\mu = (1, 0, 0, 0)$.

$$T^{\mu\nu} = \text{diagonal}\left\{\rho,\; a^{-2}\left[1-(r/r_0)^2\right] P,\; a^{-2} r^{-2} P,\; (a\, r\sin\theta)^{-2} P\right\}.$$

$$T^\mu{}_\nu = T^{\mu\alpha} g_{\alpha\nu} = \text{diagonal}(-\rho, P, P, P)$$

$$T = T^\mu{}_\mu = -\rho + 3P$$

$$T_{\alpha\beta} = g_{\alpha\mu}\, g_{\beta\nu}\, T^{\mu\nu} = \text{diagonal}\left\{\rho,\ a^2\left[1-(r/r_0)^2\right]^{-1} P,\ a^2\, r^2\, P,\ (a\, r \sin\theta)^2\, P\right\}$$

The source of the curvature

$$S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T$$

$$= \frac{1}{2}\ \text{diagonal}\left\{\rho + 3\,P,\ a^2\left[1-(r/r_0)^2\right]^{-1} (\rho - P),\ a^2\, r^2(\rho - P),\ (a\, r\sin\theta)^2\,(\rho - P)\right\}$$

Then

$$\boxed{S_{tt} = \frac{1}{2}\,(\rho + 3\,P)}$$

$$\boxed{S_{ij} = \frac{1}{2}\,(\rho - P)\, a^2\, \tilde{g}_{ij}}$$

Since $R_{\mu\nu} = -8\,\pi\, G\, S_{\mu\nu}$, the space-space part of the Ricci tensor is

$$\boxed{R_{ij} = f(t)\ \tilde{g}_{ij}}$$

even though it involves derivatives of the metric.

5 Apr 2012