The Schwarzschild metric

Schwarzschild’s formulation of the problem

What is the metric outside a spherically symmetric, static star? Conditions:

1. The metric does not change with time.
2. The metric is spherically symmetric.
3. The metric must be the same as Newton’s gravity far from the star.

Recall (13 Mar) we found in the Newtonian limit that

\[ g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

Here \( \phi = -GM/r \) is Newton’s gravitational potential. Note the Newtonian limit is accurate to first order in \( \phi \) for the time-time term but not the space-space term because the equation of motion picks out \( T^{\tau\tau} \). In Newton’s laws, space is not curved.

4. Assume the metric is

\[ ds^2 = -g_{\mu\nu} \, dx^\mu \, dx^\nu \]

Later we will show that this assumption is not restrictive.

5. What is E’s field equation in the space outside the star?

\[ G_{\mu\nu} = 8\pi \, T_{\mu\nu} \]

\( G_{\mu\nu} \) is called Einstein’s curvature tensor.

Contract:

\[ g^{\mu\nu} G_{\mu\nu} = -8\pi \, G \, T_{\mu\nu} \]

\( g^{\mu\nu} G_{\mu\nu} = 4 \) ?

Thus

\[ R = 1/4 \, G = -8\pi \, G \, T_{\mu\nu} \]

Therefore

\( G_{\mu\nu} = -8\pi \, G \, T_{\mu\nu} \)

In the space outside a star, the stress-energy tensor, which is

\[ T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

is zero. Therefore

\[ T^{\mu\nu} = 0 \]

Calculation in outline

The metric is

\[ g_{\mu\nu} = \begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \]

Calculate the Christoffel symbols

\[ \Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left( \partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda} \right) \]

Calculate the Ricci tensor

\[ R_{\alpha\beta} = \frac{1}{2} \left( \Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\beta\alpha} - \Gamma^\mu_{\mu\alpha} \Gamma^\mu_{\beta\mu} - \Gamma^\mu_{\mu\beta} \Gamma^\mu_{\alpha\mu} \right) \]

The condition \( R_{\alpha\beta} = 0 \) imposes constraints on \( A(r) \) and \( B(r) \).

Calculation in detail

The metric is

\[ g_{\mu\nu} = \begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \]

Christoffel symbols

\[ \Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left( \partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda} \right) \]

We show that \( \Gamma^r_\tau = 0 \)

\[ \Gamma^r_\tau = \frac{1}{2} g^{\mu\nu} \left( \partial_\nu g_{\mu\tau} + \partial_\tau g_{\mu\nu} - \partial_\mu g_{\tau\nu} \right) \]

\[ \Gamma^\tau_\rho = \frac{1}{2} \delta^\rho_\nu \delta^\tau_\mu \]

Other terms:

\[ \Gamma^{\theta\phi}_\phi = \frac{1}{2} \delta^{\theta\phi}_\phi \delta^\theta_\mu \]

\[ \Gamma^{\phi\theta}_\phi = \frac{1}{2} \delta^{\phi\theta}_\phi \delta^\phi_\mu \]

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See Hartle, pp. 546–547 for the Christoffel symbols of a spherically symmetric metric. In Hartle, \( A(r) = e^{\phi(r)} \)

\[ B(r) = e^{\phi(r)} \]

Of course, here \( \frac{1}{2} \delta^{\phi\phi}_\phi \delta^\phi_\mu = 0 \) and \( \frac{1}{2} \delta^{\phi\phi}_\phi \delta^\phi_\mu = 0 \).
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Schwarzschild’s assumption of the form of the metric

\[ ds^2 = -B(r) dt^2 + A(r) dr^2 + r^2 d\Omega^2 \]

is convenient but not fundamental.

Assume the metric does not depend on time and it depends on space \( \mathbb{S} \) and \( \mathbb{D} \) only through the spatial scalars \( \mathbb{S}, \mathbb{D}, \mathbb{D}, \) and \( \mathbb{D}, \mathbb{D}. \) The distance depends on all possible quadratic combinations of \( (dx, dy) \)

\[ ds^2 = -F(r) dt^2 + 2 E(r) dt dr + D(r) r^2 d\Omega^2 + G(r) r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

Use spherical coordinates.

\[ (x', y', z') = (r \sin \theta \cos \phi, \ r \sin \theta \sin \phi, \ r \cos \theta) \]

Then

\[ ds^2 = -F(r) dt^2 + 2 E(r) dt dr + D(r) r^2 d\Omega^2 + G(r) r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

Since the metric is static, we are free to replace \( t \) by \( \tau \) and \( t \) and \( r \) terms are

\[ -F(r)(dt' + f'(r) dt)^2 + 2 r E(r) dr dt' + D(r) r^2 d\Omega^2 + G(r) r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

If we choose

\[ -2 F(r) f'(r) = 2 E r \]

then the \( dt' \) \( dr \) term is zero, and

\[ ds^2 = -F(r) dt^2 + G(r) r^2 d\Omega^2 + C(r) r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

where \( G(r) = -2 [D(r) + E(r) / F(r)] \)

We may change the \( r \) coordinate, \( r^2 = x' \). Then

\[ ds^2 = -B(r') dt^2 + A(r') dr^2 + x'^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

where

\[ B(r') = F(r) \]

\[ A(r') = [1 + G(r') / C(r)][1 + C(2 B(r') C'(r')) \]^{-1} \]

Robertson-Walker metric

A 3-d space that is homogeneous and isotropic has a special choice of time. A choice of coordinates is \( (t, r, \theta, \phi) \) and the metric is

\[ ds^2 = -dt^2 + A(r)^2 [dt^2 - \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)] \]

\( r, \theta, \phi \) is called the comoving coordinate. A galaxy stays at the same position; time changes \( r_0^2 \) can have any value, positive or negative. \( k \) is called the expansion parameter.

Friedman’s equation is

\[ \frac{1}{A^2} \frac{dA}{dt} + \frac{G}{A} = \frac{kr^2}{(1 - kr^2)^{3/2}} \]

We derived Friedmann’s equation except for the constant \(-c_0^2\) on the RHS.
Form of the metric

Assumptions:
1) There is a time coordinate that is proper time.
2) At a given time, the space within a small bubble is isotropic.
3) At a given time, the space is homogeneous.

The metric is
\[ds^2 = -dt^2 + \sum_{i=1}^{3} dr_i^2 + \sum_{i=1}^{3} d\theta_i^2 \]
Q: What assumptions have gone into writing a metric of this form?

Since the space is homogeneous, \(\mathcal{R}(1, r_1, r_2, r_3)\) can depend on time and also on the distance between the points. It cannot depend on the location. Therefore
\[ds^2 = -dt^2 + \sum_{i=1}^{3} (dr_i)^2 \]
Q: Have we eliminated the possibility of a curved space by grouping the \(dr, d\theta, \) and \(d\phi\) terms together as \((dr^2 + r^2 d\theta^2 + \sin^2 \theta d\phi^2)\), which is the case for flat, spherical coordinates?

By mapping the 3 dimensions onto the surface of a 4-dimensional symmetric space, we can show that the possible metrics are
\[ds^2 = -dt^2 + \sum_{i=1}^{3} (dr_i)^2 + \sum_{i=1}^{3} (d\theta_i)^2 + \sin^2 \theta (d\phi)^2\]
where \(\vec{r}\) can be positive or negative.

Derivation of Friedman’s equation

The metric is
\[ds^2 = -dt^2 + \sum_{i=1}^{3} (dr_i)^2 + \sum_{i=1}^{3} (d\theta_i)^2 + \sin^2 \theta (d\phi)^2\]
Since the coordinate time is the same as proper time, in average the galaxies are at rest.

Einstein’s equation is
\[G_{\mu\nu} = \frac{8\pi}{c^4} T_{\mu\nu}\]
Plan: Calculate the curvature tensors and find conditions on \(a(t)\) and \(r_0\) that satisfy E’s equations.

Rewrite
\[ds^2 = -dt^2 + \sum_{i=1}^{3} (dr_i)^2 + \sum_{i=1}^{3} (d\theta_i)^2 + \sin^2 \theta (d\phi)^2\]
as
\[ds^2 = -dt^2 + \sum_{i=1}^{3} dr_i^2 + \sum_{i=1}^{3} d\theta_i^2 + \sum_{i=1}^{3} d\phi_i^2\]
where \(\vec{r} =\)
\[
\begin{pmatrix}
1 - (c^2 r_0) & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}
\]
Compute the Christoffel symbols

a) Time term
\[\Gamma_i{}^{0}{}_{0}{}^{a} = \frac{1}{2} g^{0b} (\partial_0 a^b + \partial_a a^0 - \partial_a a^b)\]
The source of the curvature

\[ S_{\mu \nu} = T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T \]

is diagonal

\[ = \frac{1}{2} \text{ diagonal } \left[ \rho (\rho + 3 P), \rho^2 \left(1 - (\rho/\rho_0)^2\right)^{-1} \left(1 + \rho - \rho_0\right), \rho^2 \left(1 - (\rho/\rho_0)^2\right)^{-1} \left(\rho - \rho_0\right) \right] \]

Then

\[ S_0 = \frac{1}{2} (\rho + 3 P) \]

\[ S_\theta = \frac{1}{2} (\rho - P) \sqrt{\rho} \]

Since \( R_{\mu \nu} = -8 \pi G S_{\mu \nu} \), the space-space part of the Ricci tensor is

\[ R_0 = f \frac{\partial f}{\partial x_0} \]

even though it involves derivatives of the metric.

5 Apr 2012