PHY321 Homework Set 10

1. [5 pts] A survey mission to a planetoid of radius R drills a hole through the center of the planetoid. Within the hole the gravitational acceleration is measured to vary with the distance r from the center as

$$g(r) = -\frac{g_0}{1-\alpha}\frac{r}{R}\left(1 - \alpha\frac{r}{R}\right),$$

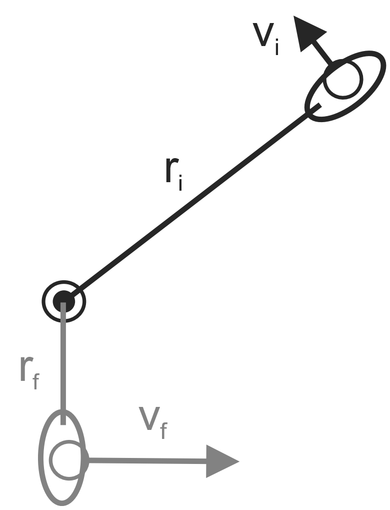
where g_0 and α are constants, $g_0 > 0$ and $0 < \alpha < 1$.

   (a) Assuming that the mass within the planetoid is distributed in a spherically symmetric manner, use the differential form of the Gauss' law to determine the mass density $\rho(r)$ for the planetoid. Is there any additional condition that α must meet to ensure physical results?

   (b) Integrate the density to find the net mass M of the planetoid.

   (c) Check your result for M by using the integral form of the Gauss' law, applied to the planetoid's surface.

2. [5 pts] A skater of mass of 55 kg moves without friction across ice at speed $v_i = 3.1\,\text{m/s}$. When at the distance of closest approach of $r_i = 3.4\,\text{m}$ to a vertical pole, the skater grabs a rope stretching from the pole. She slowly pulls herself towards the pole so that she eventually reduces her distance to the pole to $r_f = 1.7\,\text{m}$.



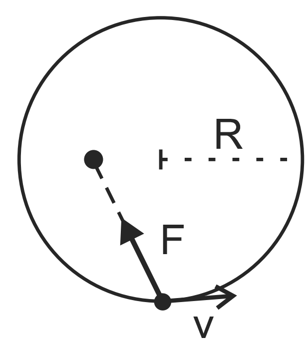
   (a) What is the angular momentum of the skater around the pole?

   (b) What is the speed v_f of the skater when at r_f from the pole?

   (c) What is the minimum work that the skater needed to carry out in pulling the rope?

   (d) Express that work in terms of the change in effective centrifugal potential energy.

3. [5 pts] A particle of unknown mass moves in a circular orbit of radius R under the influence of a central force centered at some point inside the orbit. The minimum and maximum speeds of the particle are $v_{\min}$ and $v_{\max}$ respectively. Find the orbital period T in terms of these speeds and the radius of the orbit. Hint: Use the second Kepler's law.



4. [5 pts] A satellite moves in a circular orbit in the Earth's equatorial plane. Seen from the Earth, it appears to be stationary. (a) Find the radius of its orbit. (b) Determine the minimal number of such satellites needed to ensure that at least one stationary satellite is seen from any location around the equator.

5. [5 pts] Suppose all planets move around the Sun in circular orbits under the influence of the inverse-cube rather than the inverse-square force-law, $F = \mathcal{G}mM/r^3$. Find a relation between the orbital period and the radius of the orbit for the planets in this situation, i.e. a new version of the Kepler's third law.

6. [5 pts] Determination of a trajectory $r(\theta)$ for a $1/r^4$ potential generally involves so-called elliptic integrals. In the case of an attractive potential and net energy equal to zero, though, the trajectory may be found in terms of elementary functions. Take the case of a particle of mass m, with angular momentum ℓ and energy $E = 0$, moving under an influence of the potential $U(r) = -k/r^4$, where k is a constant, $k > 0$.

   (a) Sketch the effective potential $V(r)$. How does the potential behave when $r \to 0$ and $r \to \infty$? Determine the range of r possible for $E = 0$.

   (b) Use the integral relation following from conservation laws that connects r and θ and obtain $r(\theta)$ in terms of elementary functions.

   (c) Draw the trajectory that represents the obtained dependence $r(\theta)$. Mark the center of the force in relation to the trajectory. Indicate how the trajectory is affected by the value of ℓ.